On the Cartan Invariants of Algebras

Masaru Osima*

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ON THE CARTAN INVARIANTS OF ALGEBRAS

MASARU OSIMA

1. Let $A$ be an algebra with unit element over an algebraically closed field $K$ and let

(1) \[ A = A^* + N \]

be a splitting of $A$ into a direct sum of a semisimple subalgebra $A^*$ and the radical $N$ of $A$. We shall denote by

(2) \[ A^* = A_1^* + A_2^* + \cdots + A_k^* \]

the unique splitting of $A^*$ into a direct sum of simple invariant subalgebras. Let $e_{i, \alpha}(a, \beta = 1, 2, \ldots, f(i))$ be a set of matrix units for the simple algebra $A_i^*$. We denote by $F_1, F_2, \ldots, F_k$ the distinct irreducible representations of $A$ and we set for $a$ in $A$

(3) \[ F_i(a) = (f_{i, \alpha}^i(a)). \]

Let

(4) \[ e_{i, \alpha, \beta} b_n e_{j, \alpha, \beta} \quad \alpha = 1, 2, \ldots, f(i_n) \]
\[ \beta = 1, 2, \ldots, f(j_n) \]

be the Cartan basis\(^1\) of $A$. An element $a$ of $A$, expressed in terms of the Cartan basis elements will have the form

(5) \[ a = \sum_{u, \alpha, \beta} h_{u, \alpha, \beta}^\alpha e_{i, \alpha, \beta} b_n e_{j, \alpha, \beta}. \]

For a fixed $u$, we arrange the coefficients $h_{u, \alpha, \beta}^\alpha(a)$ in a matrix $H_u(a) = (h_{u, \alpha, \beta}^\alpha(a))$. The additive group $H_u(a)$ is called an elementary module of $A$. In particular, for $b_i = e_{i, 11}$ we have $H_i(a) = F_i(a)$, that is,

(6) \[ h_{u, \alpha, \beta}^\alpha(a) = f_{i, \alpha, \beta}^i(a) \quad (i = 1, 2 \ldots, k). \]

Let $d_1, d_2, \ldots, d_n$ be a basis $(d_i)$ of $A$. Then

(7) \[ d_i = \sum_{u, \alpha, \beta} h_{u, \alpha, \beta}^\alpha(d_i) e_{i, \alpha, \beta} b_n e_{j, \alpha, \beta}. \]

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1) See Nesbitt [3], Scott [5].
or in matrix form

\[(d_s) = (e_{i_u}, a_i b_{i_a}, e_{j_u}, f_{j_s}) (h_{u\beta}^\alpha (d_s))\]

(u, α, β row index: s column index). Since \((d_s)\) is a basis of \(A\), \((h_{u\beta}^\alpha (d_s))\) is a non-singular matrix. Hence we have

**Lemma 1.** If \((d_s)\) is a basis of \(A\), then \(h_{u\beta}^\alpha (d_s)\) \((u = 1, 2, \ldots, t; \alpha = 1, 2, \ldots, f(i_u); \beta = 1, 2, \ldots, f(j_u))\) are linearly independent.

In particular, we obtain from (6)

**Lemma 2.** If \((d_s)\) is a basis of \(A\), then \(f_{u\beta}^\alpha (d_s)\) \((i = 1, 2, \ldots, k; \alpha, \beta = 1, 2, \ldots, f(i))\) are linearly independent.

We denote by \(\chi_i\) the character of \(F_i\). Then \(\chi_i(a) = \sum f_{i\beta}^\alpha (a)\). By Lemma 2

**Theorem 1.** Let \((d_s)\) be a basis of \(A\). Then \(\chi_1(d_s), \chi_2(d_s), \ldots, \chi_k(d_s)\) are linearly independent.

Now we can prove the following theorem by a procedure similar to that of Brauer and Nesbitt\(^1\).

**Theorem 2.** Let \(M_1\) and \(M_2\) be two representations of \(A\). If both \(M_1(d_s)\) and \(M_2(d_s)\) have the same characteristic roots for every \(d_s\) of a basis \((d_s)\), then \(M_1\) and \(M_2\) have the same irreducible constituents: \(M_1 \rightarrow M_2\).

2. In this section we assume that \(A\) is an algebra with unit element over an algebraic number field \(K\) and that the irreducible representations \(Z_1, Z_2, \ldots, Z_k\) of \(A\) in \(K\) are absolutely irreducible. Let \(J\) be a domain of integrity in the algebra \(A\) in the following sense\(^2\): (1) \(J\) is a subring of \(A\); (2) \(J\) contains \(n\) linearly independent elements of \(A\); (3) the elements of \(J\) when expressed by a basis \(e_1, e_2, \ldots, e_n\) of \(A\) have the form \(\sum a_i e_i\) with \(a_i = b_i / w\) where \(w\) is a fixed denominator in \(K\) and \(b_i\) are integers of \(K\); (4) \(J\) contains the ring \(\sigma\) of all integers of \(K\). Every ideal \(m\) of \(\sigma\) generates the ideal of \(J\) which may be denoted by \(m\) again. Let \(p\) be a fixed prime ideal of \(\sigma\). We denote by \(\sigma^*\) the ring of all \(p\)-integers of \(K\). Then \(\sigma^*\) and \(J\) generate a subring \(J^*\) of \(A\). \(J^*\) has a basis \(v_1, v_2, \ldots, v_n\) such that every \(\alpha\) of \(J^*\) can be written uniquely in the form

1) Cf. Brauer and Nesbitt [2], p. 3.
2) See Brauer [1].
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\[ \alpha = c_1 \beta_1 + c_2 \beta_2 + \cdots + c_n \beta_n, \quad c_i \text{ in } \mathfrak{v}^*. \]

The \( \eta_i \) can be chosen in \( J \). The ideal \( \mathfrak{v} \) generates an ideal of \( \mathfrak{v}^* \) and an ideal of \( J^* \), both of which will be denoted by \( \mathfrak{p}^* \). We denote the residue class of an element \( \alpha \) of \( J^* \) (mod \( \mathfrak{p}^* \)) by \( \bar{\alpha} \). We have

\[ \bar{\mathfrak{v}} = \mathfrak{v}^*/\mathfrak{p}^* \cong \mathfrak{v}/\mathfrak{p}^*; \quad \bar{A} = J^*/\mathfrak{p}^* \cong J/\mathfrak{p} \]

for the residue class field and residue class algebra. The elements \( \bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_n \) form a basis of \( \bar{A} \) with regard to \( \bar{\mathfrak{v}} \). Let \( S(\bar{\alpha}) \) and \( \bar{R}(\bar{\alpha}) \) be the left and the right regular representations of \( \bar{A} \), formed by means of the basis \( (\bar{\eta}_i) \). Every \( \alpha \) of \( J^* \) is then represented by matrices \( S(\bar{\alpha}) \) and \( \bar{R}(\bar{\alpha}) \) with coefficients in \( \mathfrak{v}^* \). Hence \( \bar{\alpha} \rightarrow S(\bar{\alpha}) \) and \( \bar{\alpha} \rightarrow \bar{R}(\bar{\alpha}) \) give the left and the right regular representations of \( \bar{A} \), formed by means of the basis \( (\bar{\eta}_i) \). We denote by \( F_1, F_2, \ldots, F_m \) the distinct absolutely irreducible representations of \( \bar{A} \). Let us assume here that all \( F_{\lambda} \) lie already in \( \bar{\mathfrak{v}} \). Then we have

\[ \sum_{i,j} c_{ij} Z_i(\alpha) \times Z_j(\bar{\beta}) \]

where \( c_{ij} \) and \( c_{\lambda}^* \) denote the Cartan invariants of \( A \) and \( \bar{A} \) respectively. We may assume that \( Z_i \) represents the elements of \( J^* \) by matrices with coefficients in \( \mathfrak{v}^* \). Then \( Z_{\lambda}(\bar{\alpha}) \) gives a representation of \( \bar{A} \). Let \( d_{\lambda \kappa} \) denote the multiplicity of \( F_{\lambda}(\bar{\alpha}) \) in \( Z_{\kappa}(\bar{\alpha}) \):

\[ Z_{\lambda}(\bar{\alpha}) \leftarrow \sum_{\kappa} d_{\lambda \kappa} F_{\kappa}(\bar{\alpha}), \]

The \( d_{\lambda \kappa} \) are called the decomposition numbers of \( A \).

**Theorem 3.** Let \( c_{ij}, c_{\lambda}^* \) be the Cartan invariants of \( A \) and \( \bar{A} \). Then

\[ c_{\lambda}^* = \sum_{i,j} d_{\lambda \kappa} c_{ij} d_{\kappa \lambda}, \]

where \( d_{\lambda \kappa} \) are the decomposition numbers of \( A \).

**Proof.** From (13) we have

\[
\sum_{i,j} c_{ij} Z_i(\bar{\alpha}) \times Z_j(\bar{\beta}) \leftarrow \sum_{i,j} c_{ij} \left( \sum_{\lambda} d_{\lambda \kappa} F_{\kappa}(\bar{\alpha}) \right) \times \left( \sum_{\lambda} d_{\lambda \kappa} F_{\lambda}(\bar{\beta}) \right) \\
= \sum_{\lambda, \kappa} \left( \sum_{i,j} c_{ij} d_{\lambda \kappa} \right) F_{\lambda}(\bar{\alpha}) \times F_{\kappa}(\bar{\beta}).
\]

1) See Osima [4].
By (11), \( S(\overline{a})R'(\overline{b}) \) and \( \sum_{i,j} c_{ij} Z_i(\overline{a}) \times Z_j(\overline{b}) \) have the same characteristic roots for every \( \overline{a} \) and \( \overline{b} \). Hence it follows from Theorem 2 that

\[
S(\overline{a})R'(\overline{b}) \leftrightarrow \sum_{i,j} c_{ij} Z_i(\overline{a}) \times Z_j(\overline{b}).
\]

Consequently we have from (12)

\[
\sum_{\xi,\lambda} c_{\xi,\lambda} F_\xi(\overline{a}) \times F_\lambda(\overline{b}) \leftrightarrow \sum_{\xi,\lambda} (\sum_{i,j} d_{\xi,i} c_{ij} d_{\lambda,j}) F_\xi(\overline{a}) \times F_\lambda(\overline{b}),
\]

so that we obtain

\[
c_{\xi,\lambda}^* = \sum_{i,j} d_{\xi,i} c_{ij} d_{\lambda,j}.
\]

We set \( C = (c_{ij}) \), \( D = (d_{\xi,i}) \) and \( C^* = (c_{\xi,\lambda}^*) \). Then

\[
(15) \quad C^* = D'CD.
\]

This shows that if \( C \) is a symmetric matrix, then \( C^* \) is also symmetric. If \( A \) is semisimple, then \( C \) is a unit matrix. Hence, from (15) we obtain

\[
(16) \quad C^* = D'D.
\]

**References**


**Department of Mathematics, Okayama University**

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