Signatures on a Ring

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Introduction. The purpose of this paper is to define and to investigate a
generalization of signatures. As is well known, an ordering of a field \(K\) is given
by a signature \(\sigma: K^* \to \{\pm 1\}\), or a ring homomorphism \(\sigma: W(K) \to \mathbb{Z}\). More
generally, an ordering of higher level of \(K\) is given by a character \(\chi: K^* \to S^1\)
which is called a signature of \(K\) in [4]. We give a generalization of such
signatures, and a general theory of signatures of a ring. Let \(R\) be any ring with
identity 1, and \(F\) a field-like semigroup (simply called \(f\)-semigroup) with zero
element 0, unit element 1 and a unique element \(-1\) of order 2, which is defined in
§1. A signature of \(R\) over \(F\) is defined as a map \(\sigma: R \to F\) satisfying
conditions \(\sigma(-1) = -1\), \(\sigma(ab) = \sigma(a)\sigma(b)\), and \(\sigma(a+b) = \sigma(b)\) providing \(\sigma(a) = 0\) or \(\sigma(a) = \sigma(b)\), which includes notions of orderings or higher level orderings
of a field. Indeed, for a field \(R\), if one takes \(F = GF(3) = \{0, 1, -1\}\), the
signature \(\sigma\) gives an ordering of the field \(R\). If one takes \(F = \{0\} \cup S^1\), the
signature \(\sigma\) coincides with one in [4]. This definition is motivated by the results
of Craven [5], [6], [7], and Becker [2], [3]. In §1, we introduce a topology on
the set \(X(R, F)\) of all signatures of \(R\) over \(F\), which is a generalization of the
space of “real spectrum” in [8] and “space of orderings” in [16]. In §3, under
the assumption that \(R\) is a commutative ring and \(F\) is a finite-\(f\)-semigroup, it is
proved that for the quotient ring \(S^{-1}R\) by a multiplicatively closed set \(S\) or \(R\),
the topological space \(X(S^{-1}R, F)\) is homeomorphic to a subspace \(X^f(R, F)\) of
\(X(R, F)\), and for a semilocal commutative ring \(R\), that there is a one to one
correspondence between the set of infinite primes of level 1 and the set of ring
homomorphism of the Witt ring \(W(R)\) onto the integers. Throughout this paper,
we assume that every ring has identity 1, every ring homomorphism maps 1 to
1, and the unit group of the ring \(R\) is denoted by \(R^*\). Furthermore, the number
of elements of a finite set \(F\) is denoted by \(|F|\), and for sets \(A\) and \(B\), \(A \setminus B := \{a \in A \mid a \notin B\}\).

1. Signatures over any \(f\)-semigroup. Let \(R\) be any (non-commutative)
ring with identity 1.

Definition. A multiplicative abelian semigroup \(F\) with unit element 1 and
zero element 0, i.e. \(x1 = 1x = x, x0 = 0x = 0\) for \(\forall x \in F\), is called an \(f\)-semigroup (field-like semigroup), if the subset \(F^* = F \setminus \{0\}\) is a group with a
unique element \(-1\) of order 2.

Any field with characteristic not 2 is an f-semigroup.

**Definition.** Let \(F\) be an f-semigroup. A map \(\sigma: R \to F\) is called a signature of \(R\) over \(F\), if it satisfies the following conditions:

1) \(\sigma(-1) = -1\),
2) \(\sigma(ab) = \sigma(a)\sigma(b)\) for all \(a, b \in R\),
3) either \(\sigma(a) = 0\) or \(\sigma(a) = \sigma(b)\) implies \(\sigma(a+b) = \sigma(b)\).

By [10], a subset \(P\) of \(R\) which is closed under the addition and multiplication of \(R\), and which does not contain \(-1\), is called a preprime, and a maximal preprime is called a prime of \(R\). A preprime containing 1 will be called an infinite preprime, and a maximal infinite preprime of \(R\). Furthermore, an infinite preprime \(P\) will be called an infinite quasiprime, if \(P \cap -P\) is a two sided ideal of \(R\) such that \(R/(P \cap -P)\) is an integral domain.

**Notation.** For a signature \(\sigma: R \to F\) of \(R\) over \(F\) and \(a \in F\), we denote by \(F^*\), \(G(\sigma)\), \(\mathcal{V}_\sigma(a)\) and \(P(\sigma)\) the following sets; \(F^* = F \setminus \{0\}\), \(G(\sigma) = \text{Im} \sigma \cap F^*\), \(\mathcal{V}_\sigma(a) = \{r \in R \mid \sigma(r) = a\}\) and \(P(\sigma) = \mathcal{V}_\sigma(a) \cup \mathcal{V}_\sigma(-a)\).

**Proposition 1.1.** Let \(\{\mathcal{V}_\alpha \mid \alpha \in F\}\) be a family of subsets of \(R\). There exists a signature \(\sigma: R \to F\) of \(R\) over \(F\) with \(\mathcal{V}_\sigma(\alpha) = \mathcal{V}_\alpha\) for all \(\alpha \in F\), if and only if the following conditions hold:

1) \(R = \bigcup_{\alpha \in F} \mathcal{V}_\alpha\), and \(\mathcal{V}_\alpha \cap \mathcal{V}_\beta = \phi\) for \(\alpha \neq \beta\) in \(F\),
2) \(-1 \in \mathcal{V}_{-1}\),
3) \(\mathcal{V}_a \mathcal{V}_b \subseteq \mathcal{V}_{ab}\), if \(\mathcal{V}_a \neq \phi\) and \(\mathcal{V}_b \neq \phi\),
4) \(\mathcal{V}_a + \mathcal{V}_b \subseteq \mathcal{V}_a\) and \(\mathcal{V}_a + \mathcal{V}_b \subseteq \mathcal{V}_b\) for \(\mathcal{V}_a \neq \phi\).

The proof is immediately from the definition of signature.

**Corollary 1.2.** Let \(\sigma: R \to F\) be a signature.

1) \(\mathcal{V}_\sigma(a) = P(\sigma) \cap -P(\sigma)\) is a prime ideal, and \(P(\sigma)\) is an infinite quasiprime of \(R\). \(G(\sigma)\) is a subsemigroup of \(F^*\) containing \(-1\).
2) If \(G(\sigma)\) is a finite set, then it is a group with even order.

**Lemma 1.3.** Let \(\sigma\) and \(\tau\) be signatures of \(R\) over \(F\), and assume that \(G(\sigma)\) is a subgroup of \(F^*\).

1) If \(\mathcal{V}_\sigma(a) \subseteq \mathcal{V}_\tau(a)\), then the following conditions are equivalent:
   1) \(\mathcal{V}_\sigma(a) = \mathcal{V}_\tau(a)\),
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2) \( \mathfrak{V}_a(\sigma) \cap \mathfrak{V}_a(\tau) = \emptyset \) for some \( a \in G(\sigma) \).
3) \( \mathfrak{V}_a(\sigma) \cap \mathfrak{V}_a(\tau) = \emptyset \) for all \( a \in G(\sigma) \).

Suppose \( P(\sigma) \subseteq P(\tau) \). Then \( \mathfrak{V}_a(\sigma) = \mathfrak{V}_a(\tau) \) if and only if \( \mathfrak{V}_a(\sigma) \subseteq \mathfrak{V}_a(\tau) \).

Proof. (1) \( \implies 3 \) and (3) \( \implies 2 \) are easy.

(2) \( \implies 1 \): Suppose \( \mathfrak{V}_a(\sigma) \neq \mathfrak{V}_a(\tau) \), then \( H = \{ a \in G(\sigma) \mid \mathfrak{V}_a(\sigma) \cap \mathfrak{V}_a(\tau) \neq \emptyset \} \) is a non-empty subset of \( G(\sigma) \). By (3) in (1.1), it follows that \( a \in H \) and \( \beta \in G(\sigma) \) imply \( a\beta \in H \). Since \( G(\sigma) \) is a group we get \( H = G(\sigma) \).

(2) is easy from (1).

Definition. By \( X(R, F) \), we denote the set of all signatures of \( R \) over \( F \). On the \( f \)-semigroup \( F \), we can define a topology such that \( \{0\} \) is a closed subsets and \( \{a\} \) is an open subset of \( F \) for every \( a \in F^* \). For the discrete space \( R \), the power space \( F^\# \) has an open base consisting of \( \{ f \in F^\# \mid f(a_i) \in U_i ; i = 1, 2, \ldots, n \} \) for every finite subset \( \{a_1, a_2, \ldots, a_n\} \) of \( R \) and any open subsets \( U_1, U_2, \ldots, U_n \) of \( F \). We introduce a topology on \( X(R, F) \) as a subspace of \( F^\# \).

Proposition 1.4. The topological space \( X(R, F) \) has the following properties:

1) If \( F \) is a finite set, then \( X(R, F) \) is a compact space.
2) For any \( a \in R \) and \( a \in F^* \), \( H_a(a) = \{ \sigma \in X(R, F) \mid \sigma(a) = a \} \) is an open subset of \( X(R, F) \). The finite intersections of \( H_a(a) \)'s for \( a \in R \) and \( a \in F^* \) form an open basis of \( X(R, F) \), so \( X(R, F) \) is a \( T_0 \)-space.
3) For any \( a \in R \) and \( a \in F \), \( H_a(a) = \{ \sigma \in X(R, F) \mid \sigma(a) = 0 \} \) and \( H^*(a) = H(a) \cup H_a(a) \) are closed subset of \( X(R, F) \).

Proof. (1): Suppose \( |F| < \infty \). By \( F_a \), we denote the discrete space on the set \( F \) in order to distinguish from the above topology on \( F \). It is easy to see that \( X(R, F) \) is a closed subset of \( (F_a)^\# \). Since \( X(R, F) \) is a compact subspace of \( (F_a)^\# \) which is compact by Tychonoff's theorem, so is the subspace \( X(R, F) \) of \( F^\# \) which is the image of the continuous identity map \( 1 : (F_a)^\# \to F^\# \).

(2): From the definitions of the topology on \( F \) and \( F^\# \), it follows that the subsets \( H_a(a) \) for \( a \in R \) and \( a \in F^* \) form a subbasis of open sets in \( X(R, F) \). Suppose \( \sigma \neq \tau \) in \( X(R, F) \). There is an \( a \in F^* \) with \( \mathfrak{V}_a(\sigma) \neq \mathfrak{V}_a(\tau) \), and there exists an element \( a \in R \) such that either \( a \in \mathfrak{V}_a(\sigma) \) with \( a \notin \mathfrak{V}_a(\tau) \) or \( a \in \mathfrak{V}_a(\tau) \) with \( a \notin \mathfrak{V}_a(\sigma) \), that is, \( H_a(a) \) is an open subset of \( X(R, F) \) such that either \( \sigma \in H_a(a) \) with \( \tau \notin H_a(a) \) or \( \tau \in H_a(a) \) with \( \sigma \notin H_a(a) \). Hence, \( X(R, F) \) is \( T_0 \)-space.

(3): Since \( X(R, F) = \bigcup_{a \in F} H_a(a) \) for any \( a \in R \), it follows that \( H_a(a) \) and
H^a(a) are closed subsets in X(R, F).

**Notation.** By $F$, we denote the category of f-semigroups in which morphism $f : F_1 \to F_2$ satisfies $f(-1) = -1$, $f(0) = 0$ and $f(xy) = f(x)f(y)$ for any $x, y \in F_1$. By $R$ and $T$, we denote the category of rings with identity 1, and the category of topological spaces, respectively.

**Proposition 1.5.** For any morphisms $f : F_1 \to F_2$ in $F$ and $g : R_2 \to R_1$ in $R$, map $X(g, f) : X(R_1, F_1) \to X(R_2, F_2)$: $\sigma \mapsto f \cdot \sigma \cdot g$ is continuous, so $X(-, -) : R^* \times F \to T$ is a functor.

**Proof.** Let $f : F_1 \to F_2$ and $g : R_2 \to R_1$ be morphisms in categories $F$ and $R$, respectively. To show that $X(g, f)$ is continuous, it is sufficient to show that for any $\sigma \in X(R_1, F_1)$ and an open subset $H_{a'}(a')$ containing $X(g, f)(\sigma) (= f \cdot \sigma \cdot g)$ of $X(R_2, F_2)$, $X(g, f)^{-1}(H_{a'}(a'))$ is an open subset of $X(R_1, F_1)$. Since $f \cdot \sigma \cdot g(a') = a' \neq 0$ and $f^{-1}(a') \subseteq F^*$. $X(g, f)^{-1}(H_{a'}(a')) = \bigcup_{r \in f^{-1}(a')} H_{a'}(a')$ is an open subset of $X(R_1, F_1)$.

**Theorem 1.6.** Let $a$ be a two sided ideal of $R$ and $\psi_a : R \to R/a$; $r \mapsto [r] = r + a$ the canonical ring homomorphism. Then, $X_\sigma(R, F) : = \{ \tau \in X(R, F) | a \subseteq \psi_\sigma(\tau) \}$ is a closed subset of $X(R, F)$, and the map $X(\psi_a, I)$ induces a homeomorphism $X(R/a, F) \cong X_\sigma(R, F)$.

**Proof.** For any $\sigma \in X(R, F) \setminus X_\sigma(R, F)$, $a \notin \psi_\sigma(\sigma)$ and there is an $a \in a$ with $a \notin \psi_\sigma(\sigma)$, hence $\sigma \in H_{a\sigma}(a) \cap H_{\sigma\sigma}(a) \cap X_\sigma(R, F) = \emptyset$, so $X_\sigma(R, F)$ is a closed subset of $X(R, F)$. For any $\sigma \in X_\sigma(R, F)$, a signature $[\sigma] : R/a \to F$ is naturally defined by $[\sigma]([r]) = \sigma(r)$ for $[r] \in R/a$, because of $\sigma(a + r) = \sigma(r)$ for all $a \in a$ ($\subseteq \psi_\sigma(\sigma)$). Hence, $X(\psi_a, I) : X/(R/a, F) \to X_\sigma(R, F)$ is a bijection, and is a homeomorphism, because of $X(\psi_a, I)(H_\sigma([r])) = X_\sigma(R, F) \cap H_\sigma(r)$ for any $r \in F$ and $a \in F^*$.

**Notation.** For any $\sigma \in X(R, F)$, we use notations $\phi_\sigma \text{ and } X_\sigma(R, F)$ instead of $\psi_{\sigma \psi_\sigma}(R, F)$ and $X_{\sigma \psi_\sigma}(R, F)$, i.e. $\phi_\sigma : R \to R/\psi_\sigma(\psi_\sigma(\sigma); r \mapsto [r] = r + \psi_\sigma(\sigma)$ and $X_\sigma(R, F) : = \{ \tau \in X(R, F) | \psi_\sigma(\sigma) \subseteq \psi_\sigma(\tau) \}$, respectively.

**Corollary 1.7.** For any $\sigma \in X(R, F)$, $X_\sigma(R, F)$ is a closed subset of $X(R, F)$, and $X(\phi_\sigma, I) : X(R/\psi_\sigma(\sigma), F) \to X_\sigma(R, F)$ is a homeomorphism.

**Remark 1.8.** (1) For $\sigma \in X(R, F)$, $\sigma(R^*)$ is a subgroup of $F^*$, and $\sigma(R^*)$
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\( \subseteq G(\sigma) \). If \( u \in R^* \) and \( \sigma(u) = \alpha \), then \( u^{-1} \in \mathcal{S}_\sigma^{-1}(\sigma) \) and \( \mathcal{S}_\sigma(\sigma) = u\mathcal{S}_1(\sigma) = \mathcal{S}_1(\sigma)u \).

(2) For any \( u \in R^* \), \( H_0(u) = \emptyset \) and \( H_*(u) = H_*(u) \) is an open subset of \( X(R, F) \) for all \( \alpha \in F^* \). If \( R \) is a division ring, then \( X(R, F) \) is Hausdorff and totally disconnected. Furthermore, if \( |F| < \infty \), then \( X(R, F) \) is a Boolean space, i.e. a totally disconnected, compact and Hausdorff space.

(3) For any \( a, b \in R \) and \( \alpha, \beta \in F \), the following equalities and inequalities hold:

1. \( H_*(\alpha) = H_*(-\alpha) \) and \( H_*(\alpha) \cap H_*(\alpha) = H_0(\alpha) \).
2. \( H_*(\alpha) \cap H_*(\beta) \subseteq H_*(\alpha \beta) \) and \( H_*(\alpha) \cap H_*(\beta) \subseteq H_*(\alpha + \beta) \).
3. \( H_0(\alpha \beta) = H_0(\alpha), H_0(0) = H_1(1) = X(R, F) \) and \( H_0(1) = H_*(1) = \phi \) for all \( \alpha \in F^* \) with \( \alpha \neq 1 \).

From (1.8), the following proposition immediately follows:

**Proposition 1.9.** For any \( \sigma \in X(R, F) \), the conditions (1) and (2) are equivalent, and (1) \( \implies \) (3). If \( R \) is commutative, then the converse (3) \( \implies \) (1) holds.

1. \( \sigma(R^*) = G(\sigma) \).
2. For any \( \alpha \in G(\sigma) \), there is an \( u \in R^* \) with \( \mathcal{S}_\sigma(\sigma) = u\mathcal{S}_1(\sigma) = \mathcal{S}_1(\sigma)u \).
3. \( R^*/(R^* \cap \mathcal{S}_1(\sigma)) \cong G(\sigma) \) and \( \mathcal{S}_\sigma(\sigma)\mathcal{S}_\sigma(\sigma) = \mathcal{S}_\sigma(\sigma) \) for every \( \alpha, \beta \in G(\sigma) \).

2. **Signature over a finite f-semigroup.** In this section, we assume that \( F \) is a finite set, and deal with signatures \( \sigma : R \to F \) of \( R \) over a finite f-semigroup \( F \) so that \( G(\sigma) \) is a finite group with an even order.

**Lemma 2.1.** For any \( \sigma \in X(R, F) \) and a prime ideal \( \mathcal{P} \) of \( R \) with an integral residue domain \( R/\mathcal{P} \), it follows that

1. \( G_0(\sigma) := \{ \alpha \in G(\sigma) \mid \mathcal{P}_\sigma(\sigma) \not\subseteq \mathcal{P} \} \) is a subgroup of \( G(\sigma) \), so is \( G_1(\sigma) : = \{ \alpha \in G(\sigma) \mid \mathcal{P}_\sigma(\sigma) \not\subseteq \mathcal{P}_\sigma(\tau) \} \) for any \( \tau \in X(R, F) \), and
2. \( \mathcal{P}_\sigma(\sigma) \not\subseteq \mathcal{P} \) (resp. \( \mathcal{P}_\sigma(\sigma) \not\subseteq \mathcal{P}_\sigma(\tau) \)) implies \( G_0(\sigma) = G(\sigma) \) (resp. \( G_1(\sigma) = G(\sigma) \)).

**Proof.** (1) follows from that \( |F| < \infty \) and \( R/\mathcal{P} \) is an integral domain.

(2): For any \( \alpha \in G(\sigma), \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P} \) implies that \( \mathcal{P}_\sigma(\sigma) + \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P} \) and \( \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P} \), so (2) follows.

**Proposition 2.2.** For \( \sigma, \tau \in X(R, F) \) with \( P(\sigma) \subseteq P(\tau) \), there is a group
epimorphism \( f : G_\tau(\sigma) \to G(\sigma) \) such that
\[
\mathcal{P}_\sigma(\tau) \subseteq \bigcup_{\alpha \in \mathcal{P}_\tau(\sigma)} \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P}_\sigma(\tau) \cup \mathcal{P}_\rho(\tau)
\]
for all \( \beta \in G(\tau) \).

Proof. A map \( f : G_\tau(\sigma) \to G(\sigma) \) can be defined by making the value \( f(a) = \tau(a) \) with \( a \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) for \( a \in G_\tau(\sigma) \). Because, for \( a \in G_\tau(\sigma) \) and any \( a, a' \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \), we have \( 1 = 1(ba) = \tau(b) \tau(a) \) and \( 1 = \tau(ba') = \tau(b) \tau(a') \) for any \( b \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) (\( \neq \phi \)). Since \( \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) (\( \subseteq \mathcal{P}(\sigma) \setminus \mathcal{P}(\tau) \subseteq \mathcal{P}(\tau) \setminus \mathcal{P}_\sigma(\tau) = \mathcal{P}_1(\tau) \)). Since for \( a, a' \in G_\tau(\sigma), a \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) and \( a' \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) imply \( aa' \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \), we get \( f(a, a') = \tau(aa') = \tau(a) \tau(a') = f(a)f(a') \). so \( f \) is a homomorphism. For any \( \beta \in G(\tau) \), there is an \( a \in G(\sigma) \) with \( \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \neq \phi \), because of \( \mathcal{P}_\sigma(\sigma) \cap \mathcal{P}_\sigma(\tau) \subseteq \mathcal{P}_\sigma(\tau) \cap \mathcal{P}_\sigma(\tau) = \phi \). Hence, there is a \( b \in \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) \) with \( f(a) = \tau(b) = \beta \), so \( f \) is surjective. Suppose \( \beta \in G(\tau) \) and \( x \in \mathcal{P}_\tau(\sigma) \). Since \( \mathcal{P}_\sigma(\sigma) \setminus \mathcal{P}_\sigma(\tau) = \phi \), there exists an \( a \in G(\sigma) \) with \( x \in \mathcal{P}_\sigma(\sigma) \), so we get \( f(a) = \beta \) and \( \mathcal{P}_\sigma(\sigma) \subseteq \mathcal{P}_\sigma(\tau) \cup \mathcal{P}_\rho(\tau) \) for all \( a \in G_\tau(\sigma) \) with \( f(a) = \beta \).

Definition. Let \( G \) and \( H \) be groups. A partial map \( f : G \to H \) which is a homomorphism of a subgroup \( G_1 \) onto \( H \) will be called a partial epimorphism, and for \( b \in H, f^{-1}(H) \) and \( f^{-1}(b) \) denote subsets \( f^{-1}(H) := G \), \( f^{-1}(b) := \{ x \in G_1 \mid f(x) = b \} \) of \( G \).

**Theorem 2.3.** Let \( \sigma \) and \( \tau \) be elements of \( X(R, F) \). \( P(\sigma) \subseteq P(\tau) \) holds if and only if there is a partial epimorphism \( f : G(\sigma) \to G(\tau) \) with \( \mathcal{P}(\sigma) \subseteq \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \subseteq \mathcal{P}(\tau) \cup \mathcal{P}(\rho) \) for every \( \beta \in G(\tau) \).

**Proof.** By (2.2), the "only if" part is proved. Suppose that \( f : G(\sigma) \to G(\tau) \) is a partial epimorphism and \( \mathcal{P}(\sigma) \subseteq \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \subseteq \mathcal{P}(\tau) \cup \mathcal{P}(\rho) \) for every \( \beta \in G(\tau) \). Then we have \( \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \subseteq \bigcup_{\mu \in \mathcal{P}(\tau)} (\bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma)) \subseteq \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \), and so \( \mathcal{P}(\sigma) \subseteq \mathcal{P}(\tau) \). Since \( \mathcal{P}_1(\sigma) \subseteq \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \subseteq \mathcal{P}(\tau) \cup \mathcal{P}(\rho) \), we get \( P(\sigma) = \mathcal{P}(\sigma) \cup \mathcal{P}_1(\sigma) \subseteq \mathcal{P}(\tau) \).

**Corollary 2.4.** Let \( \sigma \) and \( \tau \) be elements of \( X(R, F) \).

1. \( P(\sigma) \subseteq P(\tau) \) and \( \mathcal{P}(\sigma) = \mathcal{P}(\tau) \) hold, if and only if there is an epimorphism \( f : G(\sigma) \to G(\tau) \) with \( \mathcal{P}(\sigma) = \bigcup_{\mu \in \mathcal{P}(\tau)} \mathcal{P}(\sigma) \) for every \( \beta \in G(\tau) \).
2. \( P(\sigma) \subseteq P(\tau) \) and \( |G(\sigma)| = |G(\tau)| \) hold, if and only if there is an isomorphism \( f : G(\sigma) \to G(\tau) \) with \( \mathcal{P}(\sigma) = \mathcal{P}(\tau) \) for every
\( a \in G(\sigma). \)

(3) \( P(\sigma) = P(\tau) \) holds, if and only if there is an isomorphism \( f : G(\sigma) \rightarrow G(\tau) \) with \( \tau = f \cdot \sigma \) (i.e. \( \mathcal{P}_a(\sigma) = \mathcal{P}_{f(\sigma)}(\tau) \) for every \( a \in G(\sigma). \))

**Proof.** (1) : Suppose \( P(\sigma) \subseteq P(\tau) \) and \( \mathcal{P}_0(\sigma) = \mathcal{P}_0(\tau). \) By (2.2), there is an epimorphism \( f : G_1(\sigma) \rightarrow G(\tau) \) with \( \mathcal{P}_a(\tau) \subseteq \bigcup_{a \in f^{-1}(\beta)} \mathcal{P}_a(\sigma) \subseteq \mathcal{P}_0(\tau) \cup \mathcal{P}_a(\tau) \) for all \( \beta \in G(\tau). \) The identity means that \( \mathcal{P}_a(\tau) = \bigcup_{a \in f^{-1}(\beta)} \mathcal{P}_a(\sigma) \) and \( G_1(\sigma) = G(\sigma). \) The converse is easy by (2.3).

(2) follows from that a partial epimorphism \( f : G(\sigma) \rightarrow G(\tau) \) with \( |G(\sigma)| = |G(\tau)| \) is an isomorphism.

(3) is easy from (1) and (2).

**Definition.** On \( X(R, F), \) we can define an equivalent relation \( \sim \) as follows: For \( \sigma, \tau \in X(R, F), \) \( \sigma \sim \tau \) if and only if there is an isomorphism \( f : G(\sigma) \rightarrow G(\tau) \) with \( \tau = f \cdot \sigma, \) i.e. \( P(\sigma) = P(\tau). \) By \( X^*(R, F), \) we denote the quotient set \( X(R, F)/\sim. \) Then, we can identify \( X^*(R, F) \) with the sets \( P(\sigma) \) for all \( \sigma \in X(R, F), \) and introduce the Zariski topology on \( X^*(R, F), \) that is, the finite intersection of \( D(a) := \{ P(\sigma) \mid a \notin P(\sigma), \sigma \in X(R, F) \} \) for all \( a \in R. \)

**Proposition 2.5.** The map \( P(-) : X(R, F) \rightarrow X^*(R, F); \) \( \sigma \sim \rightarrow P(\sigma) \) is a continuous map, so \( X^*(R, F) \) is a compact space.

**Proof.** For any \( a \in R, \) \( P^{-1}(D(a)) = \{ \sigma \in X(R, F) \mid \sigma(a) \neq 0, 1 \} = \bigcup_{a \in G(\sigma) - 1} H_0(\sigma), \) so \( P^{-1}(D(a)) \) is an open subset of \( X(R, F). \)

**Definition.** A subset \( Y \) of \( X(R, F) \) is said to be irreducible, if for any closed subset \( A \) and \( B \) of \( X(R, F), \) \( Y \subseteq A \cup B \) implies either \( Y \subseteq A \) or \( Y \subseteq B. \)

The following lemma is immediately obtained from the above definition:

**Lemma 2.6.** A subset \( Y \) of \( X(R, F) \) is irreducible, if and only if for any \( H_0(a_1), H_0(a_2), \ldots, H_0(a_n), \) \( Y \subseteq \bigcup_{i=1}^{n} H_0(a_i) \) implies \( Y \subseteq H_0(a_i) \) for some \( i. \)

**Theorem 2.7.** If \( Y \) is a non-empty irreducible subset of \( X(R, F) \), there exists a \( \sigma \in X(R, F) \) such that the closure \( \text{Cl}(\{\sigma\}) \) of \( \{\sigma\} \) coincides with the closure \( \text{Cl}(Y) \) of \( Y, \) and the following identities hold : \( P(\sigma) = \bigcap_{\tau \in Y} P(\tau) \) and \( \mathcal{P}_a(\sigma) = (\bigcap_{\tau \in Y} \mathcal{P}_a(\tau)) \setminus \mathcal{P}_a(\sigma) \) for \( a \in F^*. \)
Proof. Let $Y$ be a non-empty irreducible subset of $X(R, F)$. We set $\mathcal{S}_a(Y) := \cap_{r \in R} \mathcal{S}_a(r) (\subseteq \{a \in R \mid Y \subseteq H_o(a)\})$ and $\mathcal{S}_a(Y) := (\cap_{r \in R} \mathcal{S}_a(r) \cup \mathcal{S}_a(Y)) \setminus \mathcal{S}_a(Y) (\subseteq \{a \in R \mid Y \not\subseteq H_o(a)\})$ for $a \in F^*$. Then, we have $\mathcal{S}_a(Y) \cup \mathcal{S}_a(Y) = \{a \in R \mid Y \subseteq H_o(a)\}$ for every $a \in F^*$. It is easy to check the conditions of (1.1) for the set $\{\mathcal{S}_a(Y) \mid a \in F\}$. But, we try only to check for conditions 1) and 3).

1): To show $R = \bigcup_{a \in F} \mathcal{S}_a(Y)$, suppose $a \in R \setminus \mathcal{S}_a(Y)$. Put $\{a, a_2, \ldots, a_r\} = \{\tau(a) \mid \tau \in Y\} \cap F^* (\neq \emptyset)$, then it means $Y \subseteq H_o(a) \cup H_o(a_2) \cup \cdots \cup H_o(a_r)$, so $Y \subseteq H_o(a_i)$ for some $a_i$, since $Y$ is irreducible. Hence, we get $a \in \mathcal{S}_a(Y)$.

3): To show $\mathcal{S}_a(Y) \cap \mathcal{S}_b(Y) \subseteq \mathcal{S}_{ab}(Y)$, suppose $\mathcal{S}_a(Y) \neq \emptyset$ and $\mathcal{S}_b(Y) \neq \emptyset$. If $a \neq 0$, either $a = 0$ or $b = 0$, so $\mathcal{S}_a(Y) \cap \mathcal{S}_b(Y) \subseteq \mathcal{S}_a(Y) (= \mathcal{S}_{ab}(Y))$ holds. Suppose $a \neq 0$. If $a \in \mathcal{S}_a(Y)$ and $b \in \mathcal{S}_b(Y)$, $\tau(ab) = \tau(a) \tau(b)$ is either $a \beta$ or $b \alpha$ for every $\tau \in Y$, that is, $ab \in \mathcal{S}_a(Y) \cup \mathcal{S}_b(Y)$. If $ab \in \mathcal{S}_a(Y)$, i.e. $\tau(ab) = 0$ for all $\tau \in Y$, then $Y \subseteq H_o(ab) = H_o(a) \cup H_o(b)$, so we get either $Y \subseteq H_o(a)$ or $Y \subseteq H_o(b)$, i.e. either $a \in \mathcal{S}_a(Y)$ or $b \in \mathcal{S}_b(Y)$, which contradicts to $\mathcal{S}_a(Y) \cap \mathcal{S}_b(Y) = \emptyset$. Hence, we get $ab \notin \mathcal{S}_a(Y)$ and $ab \in \mathcal{S}_{ab}(Y)$. Thus, by (1.1) there is a signature $\sigma \in X(R, F)$ which satisfies $\mathcal{S}_a(\sigma) = \mathcal{S}_a(Y)$ for all $a \in F$. Since $\{a \in R \mid \sigma \in H_o(a)\} = \{a \in R \mid Y \subseteq H_o(a)\}$ holds for all $a \in F$, we get $C(\sigma) = C(Y)$ and $P(\sigma) = \cap_{\tau \in Y} P(\tau)$.

Definition. By $X_d(R, F)$, or simply $X_d(R)$, we denote a subspace $\{\sigma \in X(R, F) \mid |G(\sigma)| = 2\}$ of $X(R, F)$, and by $X_d(R)$ a subspace $\{P(\sigma) \mid \sigma \in X_d(R)\}$ of $X^*(R, F)$.

Proposition 2.8. (1) $X_d(R)$ is a closed subset of $X(R, F)$, so it is compact.

(2) The map $P(-) : X_d(R) \rightarrow X_d^*(R) ; \sigma \mapsto P(\sigma)$ is homeomorphism, so we may regard as $X_d(R) = X_d^*(R)$.

(3) The finite intersections of $H_1(a) \cap X_d(R)$ for $a \in R$ form an open basis of $X_d(R)$.

Proof. If $\sigma \in X(R, F) \setminus X_d(R)$, there is an $a \in R$ with $\sigma(a) \notin \{1, -1\}$, which means $\sigma \in H_o(a)$ and $H_o(a) \cap X_d(R) = \emptyset$. Thus, we get (1).

(2): It is easy to see that $P(-) : X_d(R) \rightarrow X_d^*(R)$ is a bijection. By (2.5), $P(-)$ is continuous and the image of $H_1(a) \cap X_d(R)$ by $P(-)$ is $\{P(\sigma) \in X_d^*(R) \mid \sigma(a) = 1, \sigma \in X_d(R)\} = \{P(\sigma) \in X_d^*(R) \mid -a \notin P(\sigma)\} = D(-a) \cap X_d^*(R)$ which is an open subset of $X_d^*(R)$, so $P(-) : X_d(R) \rightarrow X_d^*(R)$ is a homeomorphism.
(3) is obvious.

**Remark 2.9.** Let $\sigma$ and $\tau$ be elements in $X_2(R)$.

1. $P(\sigma) \subseteq P(\tau)$ if and only if $\mathcal{P}_1(\sigma) \supseteq \mathcal{P}_1(\tau)$.
2. $P(\sigma) \subseteq P(\tau)$ and $\mathcal{P}_0(\sigma) = \mathcal{P}_0(\tau)$ imply $\sigma = \tau$.
3. Suppose $P(\sigma) \subseteq P(\tau)$. Then, for any $a \in R$, we have that $\sigma \in H^*_1(a) \Rightarrow \tau \in H^*_1(a)$, $\sigma \in H_0(a) \Rightarrow \tau \in H_0(a)$ and $\tau \in H_1(a) \Rightarrow \sigma \in H_1(a)$. Furthermore, if $P(\sigma) \neq P(\tau)$, there exists an $r \in R$ with $\sigma \in H_1(r)$ and $\tau \notin H_1(r)$.

**Definition.** An infinite preprime $P$ with $P \cup -P = R$ will be said to be of level 1. For an infinite preprime $P$, we denote by $P^*$ a subset $P \setminus (P \cap -P)$ ($= P \setminus -P$) of $P$. If $P$ is an infinite preprime of level 1, then $(P \cap -P)$ is a two sided ideal of $R$.

The following lemma is easy:

**Lemma 2.10.** Let $P$ be an infinite preprime of level 1. Then the following conditions are equivalent:

1. $P$ is an infinite quasiprime.
2. $P^* \cdot P^* \subseteq P^*$.
3. For $x \in R$ and $y \in P^*$, either $xy \in P$ or $yx \in P$ implies $x \in P$.

**Proposition 2.11.** (1) Let $P$ be an infinite quasiprime of level 1. Then, $P^* + P \subseteq P^*$ holds. Suppose $x \in R$ and $y \in P^*$. If $xy \in P^*$ or $yx \in P^*$ (resp. $xy \in (P \cap -P)$ or $yx \in (P \cap -P)$), then $x \in P^*$ (resp. $x \in (P \cap -P)$).

2. $X^*_2(R)$ is the set of all infinite quasiprimes of level 1.

**Proof.** (1): Suppose that $P$ is an infinite quasiprime of level 1, $y \in P^* (= R \setminus -P)$ and $z \in P$. $-(y+z) \in P$ implies $-(y+z)+z \in P$, but $-y \notin P$, hence $y+z \notin P$, i.e. $y+z \in P^*$. We get $P^* + P \subseteq P^*$. If for $x \in R$, $xy \in P^*$ (resp. $xy \in P \cap -P$), then by (2.10), $x \in P$ (resp. $x \in P \cap -P$). Since $x \in P \cap -P$, implies $xy \in P \cap -P$, i.e. $xy \notin P^*$, we get that $xy \in P^*$ implies $x \in P \setminus (P \cap -P) = P^*$.

(2): For any $P(\sigma) \subseteq X^*_2(R)$, by (1.2), $P(\sigma)$ is an infinite quasiprime of level 1. Conversely, suppose that $P$ is any infinite quasiprime of level 1. (2.10), (1) in (2.11) and (1.1) mean that there is a $\sigma \in X_2(R)$ such that $\mathcal{P}_0(\sigma) = P \cap -P, \mathcal{P}_1(\sigma) = P^*$ and $\mathcal{P}_{-1}(\sigma) = -P^*$. Hence, we get $P = P(\sigma) \subseteq X^*_2(R)$. 

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Proposition 2.12. Let $Y$ be any totally ordered non-empty subset of $(X(T)(R), \subseteq)$.

1. Regarding as $Y \subseteq X(T)(R) = X_0(R) \subseteq X(R, F)$, $Y$ is irreducible subset of $X(R, F)$, and there is a $\sigma \in X_0(R)$ such that $C(\{\sigma\}) = C(Y)$ and $P(\sigma) = \cap_{\tau \in Y} P(\tau) = \text{Inf}(Y)$ in $(X(T)(R), \subseteq)$.

2. There exists the Sup(Y) in $(X(T)(R), \subseteq)$.

3. For any $\sigma \in X_0(R)$, there is a maximal element in $\{P(\tau) \in X(T)(R) | P(\sigma) \subseteq P(\tau)\}$, and there is a minimal element in $\{P(\rho) \in X(T)(R) | P(\rho) \supseteq P(\sigma)\}$.

Proof. Let $Y$ be a totally ordered subset of $(X(T)(R), \subseteq)$. Suppose $Y \subseteq H_i(a_1) \cup \cdots \cup H_i(a_r) \cup H_0(b_1) \cup \cdots \cup H_0(b_s)$. If $Y \not\subseteq H_i(a_i)$ and $Y \not\subseteq H_0(b_j)$ for every $i$ and $j$, then there exist elements $\sigma_i$ and $\tau_j$ of $Y$ such that $\sigma_i \not\in H_i(a_i)$ and $\tau_j \not\in H_0(b_j)$ for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Since $Y$ is totally ordered, there is a unique minimal element $P(\rho)$ in $P(\sigma), \cdots, P(\tau)$, $P(\tau_1), \cdots, P(\tau_s)$. By (2.9), it follows that $\rho \notin H_i(a_i)$ and $\rho \notin H_0(b_j)$ for every $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$, which is a contradiction. Hence, $Y$ is included in some $H_i(a_i)$ or $H_0(b_j)$, so $Y$ is irreducible.

2: Since $Y$ is totally ordered, it follows that $P : \cup_{\tau \in Y} P(\tau)$ is an infinite preprime of level 1, $P \cap P = \cup_{\tau \in Y} (P(\tau) \cap P(\tau)) = \cup_{\tau \in Y} \wp(\tau)$ and $R/(P \cap P)$ is an integral domain. Hence, $P$ is an infinite quasiprime of level 1 which is contained in $X(T)(R)$ by (2.11) and coincides with Sup(Y).

3 is obtained by Zorn's lemma.

Lemma 2.13. (cf. [8], (2.1)). Let $\sigma$ and $\tau$ be elements in $X_0(R)$.

1. $P(\sigma) \not\subseteq P(\tau)$ and $P(\tau) \not\subseteq P(\sigma)$ imply $\wp(\sigma) \cap \wp^-(\tau) \neq \phi$.

2. If there exists a $\rho \in X_0(R)$ with $P(\rho) \subseteq P(\sigma) \cap P(\tau)$, either $P(\sigma) \subseteq P(\tau)$ or $P(\tau) \subseteq P(\sigma)$ holds.

3. A set $\{P(\rho) \in X(T)(R) \mid P(\sigma) \subseteq P(\rho)\}$ has a unique maximal element with respect to the ordering $\subseteq$.

Proof. (1) Suppose $P(\sigma) \not\subseteq P(\tau)$ and $P(\tau) \not\subseteq P(\sigma)$, then there are $a \in P(\sigma) \setminus P(\tau)$ and $b \in P(\tau) \setminus P(\sigma)$ which mean $a \in \wp(\tau) \cap P(\sigma)$ and $b \in \wp(\sigma) \cap P(\tau)$. Accordingly, we get that $a - b \in (P(\sigma) + \wp(\tau)) \cap (\wp(\tau) - P(\tau)) \subseteq \wp(\sigma) \cap \wp^-(\tau)$, so $\wp'(\sigma) \cap \wp^-(\tau) = \phi$.

(2) Suppose that $P(\rho) \subseteq P(\sigma) \cap P(\tau)$, $P(\sigma) \not\subseteq P(\tau)$ and $P(\tau) \not\subseteq P(\rho)$ hold for some $\rho, \sigma, \tau \in X(T)(R)$. By (1), we get $\wp(\sigma) \cap \wp^-(\tau) = \phi$. However, it is contrary to $\wp(\sigma) \cap \wp^-(\tau) = \phi$. Therefore, it is immediate from (2) and (2.12).
**Notation.** By \( X_\sharp(R) \) and \( X_\natural(R) \), we denote \( X_\sharp(R) := \{ \sigma \in X_\natural(R) \mid P(\sigma) \) is maximal in \( (X_\sharp(R), \subseteq) \) \} and \( X_\natural(R) := \{ \sigma \in X_\natural(R) \mid P(\sigma) \) is minimal in \( (X_\sharp(R), \subseteq) \) \}.

**Remark 2.14.** If \( R \) is commutative, then \( X_\sharp(R) \) coincides with the set of infinite primes of level 1 in \( R \).

**Proposition 2.15.** (1) \( X_\sharp(R) \) and \( X_\natural(R) \) are Hausdorff spaces as subspaces of \( X_\natural(R) \).

(2) \( X_\natural(R) \) is dense in \( X_\natural(R) \), i.e. \( \text{Cl}(X_\natural(R)) = X_\natural(R) \).

(3) For any \( \sigma \in X_\natural(R) \), a subset \( \{ \tau \in X_\natural(R) \mid P(\sigma) \subseteq P(\tau) \} \) is a closed subset of \( X_\natural(R) \), so is a subset \( \{ \tau \} \) for every element \( \tau \in X_\natural(R) \).

**Proof.** (1): If \( \sigma \) and \( \tau \) are distinct elements in \( X_\sharp(R) \) (resp. \( X_\natural(R) \)), then \( P(\sigma) \not\subseteq P(\tau) \) and \( P(\tau) \not\subseteq P(\sigma) \). By (2.13), there is an \( a \in \mathcal{V}_1(\sigma) \cap \mathcal{V}_{-1}(\tau) \) which satisfies \( a = H_{\mathcal{V}_1}(\sigma) \cap H_{\mathcal{V}_{-1}}(\tau) = \phi \).

(2): For any \( \sigma \in X_\natural(R) \), by (2.12) there is a \( \rho \in X_\natural(R) \) with \( P(\sigma) \subseteq P(\rho) \). (2.9) means that for any \( a \in R \), \( \sigma \in H_{\mathcal{V}_1}(a) \) implies \( \rho \in H_{\mathcal{V}_1}(a) \), that is, \( X_\natural(R) \) is dense in \( X_\natural(R) \).

(3): For a \( \sigma \in X_\natural(R) \), we put \( Y = \{ P(\tau) \in X_\natural(R) \mid P(\sigma) \subseteq P(\tau) \} \). If \( \rho \in X_\natural(R) \setminus Y \), then by (2.9) we have \( P(\sigma) \not\subseteq P(\rho) \) and \( \mathcal{V}_1(\rho) \not\subseteq \mathcal{V}_1(\sigma) \), so there is an \( a \in \mathcal{V}_1(\rho) \setminus \mathcal{V}_1(\sigma) \). \( H_{\mathcal{V}_1}(a) \) is an open subset with \( \rho \in H_{\mathcal{V}_1}(a) \) and \( H_{\mathcal{V}_1}(a) \cap Y = \phi \), that is, \( Y \) is closed in \( X_\natural(R) \).

3. Signatures of a commutative ring. In this section, we assume that \( R \) is a commutative ring with identity 1, and \( F \) is a finite f-semigroup.

**Notation.** Let \( S \) be a multiplicatively closed subset of \( R \) such that \( 1 \in S \) and \( 0 \not\in S \). By \( S^{-1}R \), we denote the quotient ring by \( S \), and by \( \phi^S : R \to S^{-1}R \), the canonical ring homomorphism. By \( X^S(R, F) \), we denote a subset \( X^S(R, F) := \{ \sigma \in X(R, F) \mid \mathcal{V}_0(\sigma) \cap S = \phi \} \). If \( \lambda \in X(R, F) \) and \( S = R \setminus \mathcal{V}_0(\lambda) \) for the prime ideal, we denote by \( X^S(R, F) \), \( R^{(S)} \) and \( \phi^S \) instead of \( X^S(R, F) \), \( S^{-1}R \) and \( \phi^S \) that is, \( X^S(R, F) := \{ \sigma \in X(R, F) \mid \mathcal{V}_0(\sigma) \subseteq \mathcal{V}_0(\lambda) \} \), \( R^{(S)} := (R \setminus \mathcal{V}_0(\lambda))^{-1}R \) and \( \phi^S : R \to R^{(S)} \).

**Theorem 3.1.** Let \( S \) be a multiplicatively closed subset of \( R \) with \( 1 \in S \) and \( 0 \not\in S \). The map \( X(\phi^S, I) \) induces a homeomorphism of \( X(S^{-1}R, F) \) onto the subspace \( X^S(R, F) \) of \( X(R, F) \), and \( G(\pi) = G(X(\phi^S, I)(\pi)) \) holds for every \( \pi \in \).
X(S^{-1}R, F).

Proof. First, we shall show that \( \text{Im} X(\phi^s, I) = X^s(R, F) \) and \( X(\phi^s, I) \) is a bijection. For any \( \pi \in X(S^{-1}R, F) \), it is easy that \( \phi^s(\pi \cdot \phi^s) \cap S = \pi(\phi^s(S) = \phi, \pi(\phi^s(S) \cap S = \phi) \) and \( X(\phi^s, I)(\pi) = \pi \cdot \phi^s \in X^s(R, F) \). Conversely, for a \( \sigma \in X^s(R, F) \), we can define a map \( \pi : S^{-1}R \to F \) as follows: For any \( x \in S^{-1}R \), there are \( s \in S \) and \( r \in R \) with \( \phi^s(s)x = \phi^s(r) \). Then, we put \( \pi(x) = \sigma(s) \cdot \sigma(r) \), so the map \( \pi \) is well defined because of \( \phi^s(\sigma) \cap S = \phi \). It is easy to see that \( \pi \) is a unique signature satisfying \( X(\phi^s, I)(\pi) = \pi \cdot \phi^s = \sigma \). Hence, we get that \( X(\phi^s, I) : X(S^{-1}R, F) \to X^s(R, F) \) is a bijection. Since \(|F| < \infty \), it follows that \( G(\pi) = G(\pi \cdot \phi^s) \) is a finite subgroup of \( F^* \). Let \( |G(\pi)| = n \). For any \( a \in S^{-1}R \) and \( a \in G(\pi) \), there are \( s \in S \) and \( r \in R \) with \( \phi^s(s)a = \phi^s(r) \) and \( H_a(a) = H_a(\phi^s(s)a) = H_a(\phi^s(s^{-1}r)) \) hold. We get that \( X(\phi^s, I)(H_a(a)) = \{ \pi \cdot \phi^s \in X^s(R, F) | \pi(\phi^s(s^{-1}r)) = a \} = H_a(s^{-1}r) \cap X^s(R, F) \) is an open subset of \( X^s(R, F) \), hence \( X(\phi^s, I) : X(S^{-1}R, F) \to X^s(R, F) \) is a homeomorphism. If \( X(\phi^s, I)(\pi) = \pi \cdot \phi^s = \sigma \) for \( \pi \in X(S^{-1}R, F) \), it is easy to see that \( G(\pi) = G(\sigma) \).

Corollary 3.2. (1) Let \( \mathcal{P} \) be a prime ideal of \( R \), and let \( S = R \setminus \mathcal{P} \). The map \( X(\phi^s, I) \) induces a homeomorphism \( X(S^{-1}R, F) \to X^s(R, F) \).

(2) For any \( \lambda \in X(R, F) \), \( X(\phi^s, I) : X(R^{(s)}, F) \to X^s(R, F) \) is a homeomorphism.

Notation 3.3. For any \( \lambda \in X(R, F) \), \( \lambda \) belongs to \( X^s(R, F) \). By \( \lambda^* \), we denote the signature \( \lambda^* : R^{(s)} \to F \) with \( \lambda = \lambda^* \cdot \phi^s \) determined by \( \lambda \) in (3.2). Then, \( \lambda^*(\langle R^{(s)} \rangle^*) = G(\lambda^*) = G(\lambda) \) hold, (cf. (1.9)).

Remark 3.4. Let \( R \) be a semilocal ring with the maximal ideals \( m_1, m_2, \cdots, m_r \) and \( \sigma \in X(R, F) \). If \( \alpha \in \cap_{i=1}^{r} G_i(\sigma) \), then \( \mathcal{P}_\sigma(\sigma) \cap R^* \neq \phi \), that is, there is a \( u \in R^* \) with \( \mathcal{P}_\sigma(\sigma) = u \mathcal{P}_\sigma(\sigma) \), where \( G_i(\sigma) = G_{m_i}(\sigma) = \{ a \in G(\sigma) | \mathcal{P}_\sigma(\sigma) \notin m_i \} \). Hence, if \( G_i(\sigma) = G(\sigma) \) for every \( i = 1, 2, \cdots, r \), then the conditions in (1.9) hold.

Proof. Suppose \( \alpha \in \cap_{i=1}^{r} G_i(\sigma) \). Using the induction on the number \( k \) of maximal ideals \( m_1, m_2, \cdots, m_k \), we show that there is a \( u \in \mathcal{P}_\sigma(\sigma) \) with \( u \notin m_i \) for \( i = 1, 2, \cdots, k \). Put \( |G(\sigma)| = n \). If \( \mathcal{P}_\sigma(\sigma) \cap m_i = \phi \), then we may exclude such a maximal ideal \( m_i \). Hence we may suppose \( \mathcal{P}_\sigma(\sigma) \cap m_i = \pi \neq \phi \) for \( i = 1, 2, \cdots, k \), using the assumption on induction for \( m_1, m_2, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k \), we can find \( u_i \in \mathcal{P}(\sigma) \) with \( u_i \notin m_j \) for \( j = 1, 2, \cdots, i-1, i+1, \cdots, k \). If \( u_1 \)}
In the last of this section, we note a relation between $X_2(R)$ and the set of ring homomorphisms of the Witt ring $W(R)$ on to the integers $Z$. Let $W(R)$ be the Witt ring of bilinear spaces over $R$ (cf. [1], p. 19). By $\text{Sig}(R)$, we denote the set of ring-homomorphisms of $W(R)$ on to $Z$. For $a \in R^*$, $[a]$ denotes the element of $W(R)$ with its representative $\langle a \rangle$, where $\langle a \rangle$ denotes a bilinear space of rank one with value $a$ modulo $R^*$.2.

**Lemma 3.5.** For any $\lambda \in X_2(R)$, the signature $\lambda^*$, defined in (3.3), determines a ring homomorphism $\lambda^* : W(R^{(a)}) \to Z$, which is denoted by $\lambda^*$ using the same notation. Therefore, a map $\theta : X_2(R) \to \text{Sig}(R) ; \lambda \mapsto \lambda^* \cdot W(\phi^*)$ is defined.

**Proof.** For $\lambda \in X_2(R)$, $\lambda^* : R^{(a)} \to F$ is a signature with $\lambda^* \cdot \phi^* = \lambda$ and $G(\lambda^*) = G(\lambda) = \{1, -1\}$. $\mathcal{S}_1(\lambda^*)$ satisfies the following conditions; $\mathcal{S}_1(\lambda^*) + \mathcal{S}_1(\lambda^*) \subseteq \mathcal{S}_1(\lambda^*), \mathcal{S}_1(\lambda^*) \cdot \mathcal{S}_1(\lambda^*) \subseteq \mathcal{S}_1(\lambda^*), \mathcal{S}_1(\lambda^*) \cap -\mathcal{S}_1(\lambda^*) = \emptyset$ and $\mathcal{S}_1(\lambda^*) \cup -\mathcal{S}_1(\lambda^*) = R^{(a)}$. Hence, we can define an ordering on the local ring $R^{(a)}$ which determines a ring homomorphism $\lambda^* : W(R^{(a)}) \to Z$ with $\lambda^*([a]) = \lambda^*(a) (\in G(\lambda^*) = \{1, -1\} \subseteq Z)$ for every $a \in (R^{(a)})^*$ (cf. (2.2) and (2.5) in [12]).

**Proposition 3.6.** Let $\mathcal{S}$ be a prime ideal of $R$, and let $S = R \setminus \mathcal{S}$.

1. If $P$ is an infinite quasiprime of level 1 in $S^{-1}R$, then so is the inverse image $(\mathcal{S})^{-1}(P)$ in $R$.

2. Let $\lambda \in X_2(R)$ and $\mu \in \text{Sig}(R)$, and $Q = \{a \in R \mid a \in S, \mu([\mathcal{S}(a)] = 1\}$. If $Q \subseteq \mathcal{S}_1(\lambda)$, then there is an $R$-algebra homomorphism $f : S^{-1}R \to R^{(a)}$ with $\mu = \lambda^* \cdot W(f)$.

**Proof.** (1): For any infinite quasiprime $P$ in $S^{-1}R$, it is easy to see that $(\mathcal{S})^{-1}(P)$ is an infinite preprime and $(\mathcal{S})^{-1}(P \cap - P) = (\mathcal{S})^{-1}(P) \cap -(\mathcal{S})^{-1}(P)$ is a prime ideal of $R$.

(2): Since $Q$ is multiplicatively closed and $Q \cup -Q = S$, it follows that $Q^{-1}Q = \{\phi^*(a)^{-1} \cdot \phi^*(b) \in S^{-1}R \mid a, b \in Q\}$ is a positive cone of an ordering on $S^{-1}R$ defined by $\mu$. Since $Q \subseteq \mathcal{S}_1(\lambda)$, there is a natural $R$-algebra homomorphism $f : S^{-1}R \to \mathcal{S}_1(\lambda)^{-1}R = R^{(a)}$. Since $f$ carries the positive cone $Q^{-1}Q$ of
ordering on $S^{-1}R$ into the positive cone $P_1(\lambda^*)$ of ordering on $R^{(1)}$, so we get $\mu = \lambda^* \cdot W(f)$.

**Corollary 3.7.** If $\lambda, \lambda' \in X_2(R)$ with $P(\lambda) \subseteq P(\lambda')$, then there is an $R$-algebra homomorphism $g : R^{(1)} \rightarrow R^{(1)}$ with $\lambda'^* = \lambda^* \cdot W(g)$.

**Proof.** Since $P(\lambda) \subseteq P(\lambda')$ implies $\mathfrak{P}_1(\lambda') \subseteq \mathfrak{P}_1(\lambda)$, the proof is immediately from (3.6).

**Theorem 3.8.** The map $\theta : X_2(R) \rightarrow \text{Sig}(R)$ is surjective, and $\theta(X_2^*(R)) = \text{Sig}(R)$, that is, for any ring homomorphism $\mu : W(R) \rightarrow \mathbb{Z}$, there is an infinite prime $P$ of level 1 in $R$ such that $\mu([a]) = 1$ for all $a \in P \cap R^*$.

**Proof.** To show that $\theta : X_2(R) \rightarrow \text{Sig}(R)$; $\lambda \sim \rightarrow \lambda^* \cdot W(\phi^5)$ is surjective, suppose $\mu \in \text{Sig}(R)$. By Lemma 1 and Proposition 1 in [9], there is a maximal ideal $m$ of $R$, and for $S = R \setminus m$ there is a ring homomorphism $\nu : W(S^{-1}R) \rightarrow \mathbb{Z}$ with $\mu = \nu \cdot W(\phi^5)$. We put $Q' = \{ \sum_i a_i \in S^{-1}R \mid a_i \in (S^{-1}R)^* \} \cup \mathfrak{P}$. By Theorem 4.1 in [14], it follows that $Q' + Q' \subseteq Q'$, $Q' \cdot Q' \subseteq Q'$, and $\mathfrak{P} = (S^{-1}R) \setminus (Q' \cup Q')$ is a prime ideal of $S^{-1}R$ with $\mathfrak{P} + Q' \subseteq Q'$. Hence, $P = \mathfrak{P} \cup Q'$ is an infinite quasiprime of level 1 in $S^{-1}R$, and so is also $(\phi^5)^{-1}(P)$ in $R$ by (3.6). Hence, there is a $\lambda \in X_2(R)$ with $P(\lambda) = (\phi^5)^{-1}(P)$. We shall show $\theta(\lambda) = \mu$. We put $Q = \{ a \in S \mid \nu([\phi^5(a)]) = 1 \}$. From the fact that $\phi^5(Q) = Q'$ and $(\phi^5)^{-1}(Q') = \mathfrak{P}_1(\lambda)$, it follows that $Q \subseteq \mathfrak{P}_1(\lambda)$, and using (3.6), there is an $R$-algebra homomorphism $f : S^{-1}R \rightarrow R^{(1)}$ making the following diagram commute:

Thus, we get $\mu = \nu \cdot W(\phi^5) = \lambda^* \cdot W(\phi^5) = \theta(\lambda)$. In the second place, we shall show $\theta(X_2^*(R)) = \text{Sig}(R)$. For any $\mu \in \text{Sig}(R)$, we can find a $\lambda \in X_2(R)$ with $\theta(\lambda) = \mu$. By (2.13), we can also find a $\lambda' \in X_2^*(R)$ with $P(\lambda) \subseteq P(\lambda')$. By (3.7), there is an $R$-algebra homomorphism $g : R^{(1)} \rightarrow R^{(1)}$ with $\lambda'^* = \lambda^* \cdot W(g)$.
Therefore, we get $\mu = \theta(\lambda')$ by the following commutative diagram:

\[
\begin{array}{ccc}
W(R) & \xrightarrow{\mu} & Z \\
\downarrow{\mu} & & \downarrow{\lambda^*} \\
W(\phi^t) & \xleftarrow{\lambda^*} & W(R^{(tt)}) \\
\downarrow{\phi^t} & & \downarrow{\phi^t} \\
W(g) & \xleftarrow{\lambda^*} & W(R^{(tt)})
\end{array}
\]

**Corollary 3.9.** If $R$ is a semilocal ring, then the map $\theta$ induces a bijection $X^n(R) \to \text{Sig}(R)$.

**Proof.** To show that $\theta : X^n(R) \to \text{Sig}(R)$ is a bijection, we suppose that $\mu \in \text{Sig}(R)$ and $\sigma, \tau \in X^n(R)$ with $\theta(\sigma) = \theta(\tau) = \mu$. By Appendix B in [14], a subset $Q = \{\sum_{i=1}^n a_i b_i \in R | a_i, a_2, \cdots, a_n \in R^*, b_1, b_2, \cdots, b_n \in R; \mu([a_i]) = 1, \sum_{i=1}^n b_i R = R\}$ of $R$ satisfies that $Q + Q \subseteq Q, QQ \subseteq Q, 1 \in Q$, and $\mathfrak{p} = R \setminus (Q \cup -Q)$ is a prime ideal of $R$ with $\mathfrak{p} + Q \subseteq Q$. Hence, $P = \mathfrak{p} \cup Q$ is an infinite quasiprime of $R$, and there is a $\lambda \in X^n(R)$ with $P(\lambda) = P$. We can easily check that $\mathfrak{p}_1(\lambda) = Q, Q \subseteq \mathfrak{p}_1(\sigma)$ and $Q \subseteq \mathfrak{p}_1(\tau)$. Accordingly, by (2.9), we get $P(\sigma) \subseteq P(\lambda)$ and $P(\tau) \subseteq P(\lambda)$, so $P(\sigma) = P(\tau) = P(\lambda)$, that is, $\sigma = \tau$.

**References**


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(Received November 10, 1991)