An Example of an Indecomposable Module which is not Injective

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Dedicated to Professor Manabu Harada on his 60th birthday

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It is well-known that every torsionfree divisible module is injective over a commutative integral domain. First we show that this theorem can be improved by using the concept of weakly $\sigma$-injective modules over a left Ore domain. Also in [2], we show that for a non-singular module $M$ if $M$ has no nonzero injective submodule, then so does $M^\lambda$ for all index sets $\lambda$. Finally, we give an example to show that the above proposition is fualse in general.

Throughout this note $R$ is a ring with identity and modules are unitary left $R$-modules unless otherwise stated. We denote the category of modules by $R$-mod and the injective hull of a module $M$ by $E(M)$. As for terminologies and basic properties concerning torsion theories and preradicals, we refer to [3]. Let $\rho$ be a preradical. We call it stable if $T(\rho)$ is closed under essential extensions. Also the left linear topology corresponding to a left exact preradical $\rho$ is denoted by $\mathcal{D}(\rho)$. Now for two preradicals $\rho$ and $\tau$, we shall say that $\rho$ is larger than $\tau$ if $\rho(M) \supseteq \tau(M)$ for all modules $M$.

We put $\mathcal{D} = \{r \in R \mid rs \neq 0$ and $sr \neq 0$ for all $s \neq 0 \in R\}$. We call a ring $R$ left Ore if for each $r \in R$ and $s \in \mathcal{D}$ there exist $r' \in R$ and $s' \in \mathcal{D}$ such that $s'r = r's \neq 0$. Also we put $\sigma(M) = \{x \in M \mid rx = 0$ for some $r \in \mathcal{D}\}$. In general, $\sigma(M)$ is not a submodule of $M$.

However

**Proposition 1.** If $R$ is a left Ore domain, then $\sigma$ is the Goldie torsion functor $G$.

**Proof.** By [3, p. 138, Example 2], $\sigma$ is a left exact radical. By assumption, every non-zero ideal of $R$ is essential in $R$. Thus $\mathcal{D}(Z)$ is the set of non-zero ideal of $R$, where $Z$ is the singular torsion functor. Thus $\sigma \leq Z$. Conversely let $r$ be a non-zero element of $R$ and let $s + Rr$ be in $R/Rr$. Then $s's = r'r$ for some $r'$ and $s'$ in $R$ by assumption. Thus $R/Rr$ is in $T(\sigma)$ and so $Rr$ is in $\mathcal{D}(\sigma)$. Hence every non-zero ideal of $R$ belongs to $\mathcal{D}(\sigma)$. Since $\sigma$ is a radical, so is $Z$, namely, $Z = G$. Hence $\sigma = G$.  

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Definition. Let $\tau$ be a preradical. We call a module $H$ (resp. weakly $\tau$-injective) if for all exact sequences of modules $O \to A \to B \to C \to O$ with $C \subseteq T(\tau)$ (resp. $B \subseteq T(\tau)$), the functor $\text{Hom}_R(-, H)$ preserves the exactness.

Lemma 2 [1, Theorem 1.11.]. Let $\tau$ be a left exact preradical. Then the following conditions are equivalent:

(i) Every $\tau$-injective module is injective.

(ii) $\bar{\tau}$ is larger than the Goldie torsion functor $G$, where $\bar{\tau}$ is the smallest radical larger than $\tau$.

By the above lemma, every $\sigma$-injective module is injective.

Theorem 3. Let $R$ be a left Ore domain. For a module $M$, the following conditions are equivalent:

(i) $M$ is divisible and weakly $\sigma$-injective.

(ii) $M$ is injective.

Proof. (ii) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (ii). By Proposition 1, $\sigma = G$. Thus it is sufficient to show that $M$ is $\sigma$-injective by Lemma 1.2. We assume that $\sigma(E(M)/M) \neq O$. Then there exists $x = x + M$ (x is in $E(M)$ and is not in $M$) such that $rx = 0$ for some $r \neq 0$ in $R$. Since $M$ is divisible and $rx$ is in $M$, $rx = rm$ for some $m$ in $M$, namely $r(x - m) = 0$. Thus $x - m$ is in $\sigma(E(M))$. Since $M$ is weakly $\sigma$-injective, $\sigma(E(M)) = \sigma(M)$ and so $x - m$ is in $M$. Hence $x$ is in $M$. This is a contradiction. Thus $M$ is $\sigma$-injective.

Since $\sigma$ is left exact, every $\sigma$-torsionfree module is weakly $\sigma$-injective. Thus we have the following famous result:

Corollary 4. Let $R$ be a commutative integral domain and $M$ a torsionfree module. Then $M$ is injective if and only if it is divisible.

We call a ring $R$ left hereditary if every left ideal of $R$ is projective. From Theorem 3 and [3, Proposition 4.5], we have

Corollary 5. Let $R$ be a left Ore domain. Then the following conditions are equivalent:

(i) Every divisible module is weakly $\sigma$-injective.

(ii) Every divisible module is $\sigma$-injective.

(iii) Every divisible module is injective.

(iv) $R$ is left hereditary.
First we give an example of a module which is divisible but not injective.

Let \( Z \) be the ring of integers, \( \mathcal{Q} \) the field of rational numbers and \( Z_p \) the localization of \( Z \) with respect to \( p \mathbb{Z} \) for a prime number \( p \).

**Example 6.** Let \( R \) be the polynomial ring over \( Z_p \) and \( K \) the quotient field of \( R \). Then \( K/R \) is not injective.

**Proof.** We put \( I = pR + xR \). Let \( f \) be a map from \( I \) to \( K/R \) with \( f(pa_0 + ax + \cdots + anx^n) = (p/x)(a_0 + ax + \cdots + anx^n) + ((1-p)/p)(a_0 + ax + \cdots + anx^n) + R \), where \( a_i (i = 0, 1, \cdots, n) \) are in \( Z_p \). Then \( f(p) = p/x + R \) and \( f(x) = 1/p + R \). Suppose that \( K/R \) is injective. Then there exists an \( R \)-homomorphism \( g : R \to K/R \) such that \( g(a) = f(a) \) for all \( a \in I \). We put \( g(1) = k + R (k \in K) \). Then \( g(p) = pk + R = p/x + R = f(p) \) and \( g(x) = xk + R = 1/p + R = f(x) \). Since \( pk - p/x \in R \) and \( xk - x/p \in R \), \( k = \frac{p + c_0x + c_1x^2 + \cdots + c_mx^{m-1}}{px} = 1 + \frac{pd_0 + pd_1x + \cdots + pd_mx^n}{px} \) for some \( c_i, d_i \in Z_p \) (\( i = 0, 1, \cdots, m \)) and \( d_j \in Z_p \) (\( j = 0, 1, \cdots, n \)). Thus \( 1 + pd_0 = p \) and so \( d_0 = (p - 1)/p \) does not belong to \( Z_p \). This is a contradiction. Hence \( K/R \) is not injective.

If a module \( M \) is nonsingular, then \( M^d \) has no nonzero injective submodule if and only if \( M \) has no nonzero injective submodule, where \( M^d \) is a direct product of copies of \( M \) for an index set \( \Lambda \) [2, Theorem 2.9]. But this is not true for some singular module \( M \).

**Lemma 7.** Let \( R \) be a commutative integral domain with quotient field \( K \neq R \). Then the following assertions hold.

1. \( K \) is an injective \( R \)-module.
2. \((K/R)^{R-\{0\}}\) has a nonzero injective submodule.

**Proof.** (1). Since \( K \) is divisible and is in \( F(\sigma) \), it is injective. (2). We consider a correspondence \( \phi : K \to (K/R)^{R-\{0\}} \) defined by \( \phi(k) = (\cdots, k/r_{\sigma}, \cdots) \). Then \( \phi \) is an \( R \)-homomorphism and \( \text{Ker}(\phi) = \{ k \in K \mid k/r_{\sigma} \in R \text{ for all } r_{\sigma} \in R - \{0\} \} \). Clearly \( \phi \) is a monomorphism. By (1), \((K/R)^{R-\{0\}}\) has a nonzero injective submodule.

**Example 8.** Let \( R = Z_p + xQ[[x]] \), where \( Q[[x]] \) is the ring of formal power series over \( Q \) and \( K \) the quotient field of \( R \). Then \( K/R \) has no nonzero
injective submodule but \((K/R)^{R-10}\) has a nonzero injective submodule.

Proof. It is sufficient to show that \(K/R\) is indecomposable and it is not injective. First we show that \(K/R\) is indecomposable. We assume that \(K/R = A/R \oplus B/R\), where \(A\) and \(B\) are \(R\)-submodules of \(K\) containing \(R\) with \(K = A + B\) and \(A \cap B = R\). We claim that \(1/x\) belongs either to \(A\) or to \(B\). Since 
\(1/x\) is in \(A + B\), there exist \(a \in A\) and \(\beta \in B\) such that 
\(1 = ax + \beta x\). Thus \(ax = 1 - \beta x\) is in \(B\) and so \(ax\) is in \(R\). Simillary \(\beta x\) is in \(R\). Therefore \(ax = a_0 + a_1x + \cdots + a_nx^n + \cdots\) and \(\beta x = b_0 + b_1x + \cdots + b_nx^n + \cdots\) for some \(a_0\) and \(b_0\) is \(Z_p\) and \(a_i\) and \(b_i\) \((i = 1, 2, \cdots)\) in \(Q\). Since \(1 = ax + \beta x\), \(a_0 + b_0 = 1\) and \(a_i + b_i = 0\) for \(i \geq 1\). Thus either \(a_0\) or \(b_0\) is a unit. If \(a_0\) (resp. \(b_0\)) is a unit, then \(ax\) (resp. \(\beta x\)) is unit in \(R\) and so \(1/x = a(ax)^{-1}\) (resp. \(\beta(\beta x)^{-1}\)) is in \(A\) (resp. \(B\)). Thus we may assume that \(1/x\) is in \(A\). Then \(Q[[x]]\) is an \(R\)-submodule of \(A\). In fact take \(\gamma = c_0 + c_1x + c_2x^2 + \cdots\) be in \(Q[[x]]\). If \(c_0 = 0\), then \(\gamma\) is in \(R < A\). On the other hand, if \(c_0 \neq 0\), then \(c_0 = (1/x) \cdot c_0x \in A\).
Since \(c_1x + c_2x^2 + \cdots\) is in \(R\), \(\gamma\) is in \(A\) and so \(Q[[x]]\) is an \(R\)-submodule of \(A\).
Next we show that \(a/x\) belongs to \(A\) for every \(a \in A\). Indeed, let \(a/x = a' + \beta'\) \((a' \in A\) and \(\beta' \in B\)). Then \(a = x a' + x \beta'\) and so \(x \beta' = a - x a'\). Thus \(x \beta'\) is in \(A \cap B = R\). We put \(x \beta' = d_0 + d_1x + d_2x^2 + \cdots\), where \(d_0 \in Z_p\) and \(d_i \in Z\) \((i = 1, 2, \cdots)\). Then \(x \beta' = c_0/x + \sum d_i x^{i-1}\). Since \(c_0/x\) is in \(A\) and \(\sum d_i x^{i-1}\) is in \(Q[[x]]\), \(\beta'\) is in \(A\). Hence \(a/x\) is in \(A\). As is easily seen, each element of \(K\) is of the form \(h(x)/x^m\), where \(h(x)\) is in \(Q[[x]]\) and \(m\) is a non-negative integer. Thus \(K = A\), namely \(K/R\) is indecomposable. Secondly we show that \(K/R\) is not injective. We put \(I = xQ[[x]]\) and \(I_n = (x/p^n)R\) for \(n = 0, 1, 2, \ldots\).
Then we have an infinite ascending chain \(I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots\) with \(\bigcup_{n=0} I_n = I\). Note that any homomorphism \(\varphi_n\) from \(I_n\) to \(K/R\) is uniquely determined by an element \(a_n\) of \(K/R\) which can be arbitrarily chosen and by the equation \(\varphi_n(a) = a a_n\) for all \(a \in I_n\). Let \(\{a_i\}\) be a sequence of integers such that \(0 \leq a_i < p\) for all \(i \geq 0\). For each \(n\), let \(f_n = (1/x) \sum a_i p^i\) be an element of \(K\) and \(a_n = f_n + R\) an element of \(K/R\). Then we have \(a_{n+1} - a_n = (1/x) a_n + p^{n+1} + R\). Since \(a_{n+1} p\) is in \(Z\), \((x/p^n)(a_{n+1} - a_n) = 0\) for \(n = 0, 1, 2, \ldots\).
Let \(\varphi_n\) be the \(R\)-homomorphism from \(I_n\) to \(K/R\) corresponding to \(a_n\) as noted above. Then the above equations imply that the system \((I_n, \varphi_n)\) forms a direct system and \(\varphi_n I_n = I\), we have the following commutative diagram

\[
\begin{array}{c}
\xymatrix{
I_0 
\ar[r] & I_1 
\ar[r] & I_2 
\ar[r] & \cdots 
\ar[r] & I_n 
\ar[r] & I_{n+1} 
\ar[r] & \cdots 
\end{array}
\]

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\[
\begin{array}{c}
\psi_n \\
\downarrow \\
\phi_m \\
\downarrow \\
\lim I_n \\
\phi = \lim \phi_n \\
\varphi_n \\
\varphi_m \\
\varphi_m \\
\varphi = \varphi_n \\
K/R
\end{array}
\]

where \( \varphi(a) = \varphi_n(a) = a\alpha_n \) for all \( a \in I_n \).

Assume that \( K/R \) is injective. Then there exists an element \( \alpha \) of \( K/R \) such that \( \varphi(s) = s\alpha \) for all \( s \in I \). If we take \( s = x/p^n \in I_n \), then \( \varphi(s) = s\alpha = s\alpha_n = \varphi_n(s) \) and so \( (x/p^n)(\alpha - \alpha_n) = 0 \). If we put \( \alpha = f + R \), where \( f \in K \), it follows that \( (x/p^n)(f - f_n) \in R \) for all \( n \), namely, \( xf = \Sigma_{i=0}^{n-1} x_{a_i} p^i \), where \( a_0 \in Z_p \) and \( b_i \in \Omega \) for \( i \geq 1 \). Then we have \( b_n = \Sigma_{i=n}^{\infty} x_{a_i} p^i + x_{a_0} p^0 \) for some \( c_0 \in Z_p \). Thus the sequence \( \Sigma_{i=n}^{\infty} x_{a_i} p^i \) converges to \( b_0 \) with respect to the \( p \)-adic topology on \( Z_p \), that is, the \( p \)-adic number \( \Sigma_{i=n}^{\infty} x_{a_i} p^i \) represents a rational number. This is absurd because the sequence \( \{ a_i \} \) of integers can be arbitrarily chosen. It follows that \( K/R \) is not injective.

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