On Jordan Left Derivations

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Throughout the present paper, $R(\neq 0)$ will represent an associative ring with center $Z$ and $X$ a nonzero left $R$-module. Following [1], $X$ is called prime if $aRx = 0$ for $a \in R$ and $x \in X$ implies that either $x = 0$ or $aX = 0$. As is well known, $R$ is a prime ring if and only if there exists a nonzero faithful prime left $R$-module. Following [2], an additive mapping $D : R \rightarrow X$ is called a Jordan left derivation if $D(a^2) = 2aD(a)$ for all $a \in R$.

Now, let $X$ be a faithful prime left $R$-module and $p$ a prime number. Suppose that $R$ is of characteristic $p$. Then for any nonzero $a \in R$, $aR(pX) = (pa)RX = 0$, and so $pX = 0$. Conversely, suppose that $px = 0$ for some nonzero $x \in X$. Then, for any $a \in R$, $0 = aR(px) = (pa)Rx$, and so $pa = 0$. Consequently, we see that $X$ is not $p$-torsionfree, or what is the same, $pX = 0$, if and only if $R$ is of characteristic $p$.

Our present objective is to improve [2, Theorem 1.2] as follows.

**Theorem 1.** Let $R$ be a prime ring of characteristic $\neq 2$, and $X$ a nonzero left $R$-module. Suppose that $X$ is faithful and prime. If there exists a nonzero Jordan left derivation $D : R \rightarrow X$, then $R$ is commutative.

In preparation for proving our theorem, we state several lemmas.

**Lemma 1.** Suppose that $X$ is faithful and prime. Let $a, b \in R$, and $x \in X$. If (the prime ring) $R$ is of characteristic $\neq 2$ and $arbx = 0$ for all $r \in R$, then $a = 0$ or $b = 0$ or $x = 0$.

**Proof.** Obviously, $0 = a(u + v)b(u + v)x = aubvx + avbux$ for all $u, v \in R$. Replacing $v$ by $rarbr$, we have $0 = aubrbrx + ararbrbux = ararbrbux$ for all $u, r \in R$. Suppose that $x \neq 0$. Noting that $X$ is faithful and prime, we obtain $uararbrb = 0$. Since $R$ is prime, [5, Theorem] shows that either $a = 0$ or $b = 0$.

**Lemma 2.** Let $R$ be a ring of characteristic 3. If $D : R \rightarrow X$ is a Jordan left derivation, then for all $a, b, c \in R$, there holds the following:

1. $D(ab + ba) = 2aD(b) + 2bD(a)$.
2. $D(aba) = a^2D(b) - baD(a)$.
(3) \[ D(abc + cba) = (ac + ca)D(b) - baD(c) - bcD(a). \]

**Proof.** See the proof of [2, Proposition 1.1].

**Lemma 3.** Let \( R \) be a ring of characteristic 3, and \( D: R \to X \) a Jordan left derivation. Suppose that \( X \) is faithful and prime. If \( D(a) \neq 0 \) for some \( a \in R \), then \( [a, [a, b]]^3 = 0 \) for all \( b \in R \).

**Proof.** Note that \( X \) is 2-torsionfree, and see the proof of [2, Lemma A].

The next will play an essential role in the proof of Theorem 1.

**Lemma 4.** Let \( R \) be a prime ring of characteristic 3. Suppose that \( X \) is faithful and prime. If there exists a nonzero Jordan left derivation \( D: R \to X \), then \( R \) has no nonzero nilpotent elements (more precisely, \( R \) has no nonzero divisors of zero).

**Proof.** Suppose, to the contrary, that \( R \) contains a nonzero element \( a \) with \( a^2 = 0 \). Then \( aD(a) = 0 \). Now, by Lemma 2(1) and (2), we obtain
\[
2aD(ba) = 2aD(ba) + 2baD(a) = D(aba) = a^2D(b) - baD(a) = 0, \quad \text{and so} \quad aD(ba) = 0.
\]
Next by Lemma 2(3),
\[
D(ab^2a) + D(baba) = D(ab^2a + baba) = (aba + ba^2)D(b) - baD(ba) - b^2aD(a) = abaD(b).
\]
Combining this with \( D(ab^2a) = 0 \) (Lemma 2(2)) and \( D(baba) = 2baD(ba) = 0 \), we obtain \( abaD(b) = 0 \). So, linearizing this on \( b \), we obtain
\[
\text{abaD(c) + acD(b) = 0 for all } b, c \in R.
\]
Replacing \( c \) by \( ac + ca \) in (#), we obtain \( abaD(ac + ca) = 0 \), and so \( abacD(a) = 0 \) by Lemma 2(1). Hence \( D(a) = 0 \) by Lemma 1. Further, \( acD(bab) = ac(b^2D(a) - abD(b)) = 0 \) by Lemma 2(2). Now, replacing \( b \) by \( bab \) in (#), we get
\[
ababaD(c) = ababaD(c) + acD(bab) = 0. \quad \text{Hence } aD(c) = 0 \text{ by Lemma 1. Replace } c \text{ by } c^2 \text{ to get } acD(c) = 0. \text{ Linearizing this on } c, \text{ we obtain } abD(c) + acD(b) = 0. \quad \text{Furthermore, replacing } c \text{ by } ac, \text{ we have } abD(ac) = 0 \text{ for all } b \in R, \text{ and so } D(ac) = 0 \text{ by the faithfulness and the primeness of } X. \text{ Recalling that } D(a) = 0 \text{ and } aD(c) = 0, \text{ we obtain } D(ca) = D(ac + ca) = 0 \text{ by Lemma 2(1), and so } D(cba) = 0. \text{ Combining this with } D(a(bc)) = 0, \text{ we have } acD(b) = D(abc) + (cb)a = 0 \text{ by Lemma 2(1). Hence } D(b) = 0 \text{ for all } b \in R. \quad \text{But this is a contradiction.}
We are now ready to complete the proof of Theorem 1.

**Proof of Theorem 1.** In view of [2, Theorem 1.2], it suffices to consider the case that $R$ is of characteristic 3. Choose $a \in R$ such that $D(a) \neq 0$. Then $[a, [a, b]] = 0$ for all $b \in R$, by Lemmas 3 and 4, and so [4, Theorem 1] shows that $a$ is in $Z$. Thus $R = Z \cup \{a \in R \mid D(a) = 0\}$. Since $D$ is nonzero, we conclude that $R = Z$, by Brauer’s trick.

Finally in connection with Theorem 1, we shall improve [6, Theorem 2] as follows:

**Theorem 2.** Let $R$ be a prime ring of characteristic $\neq 2$. If there exists a nonzero derivation $D : R \to R$ such that $[a, [a, D(a)]] \subseteq Z$ for all $a \in R$, then $R$ is commutative.

**Proof.** In view of [6, Theorem 2], it suffices to consider the case that $R$ is of characteristic 3. Then, for any $a \in R$,

\[
D(a^3) = a^3D(a) + aD(a)a + D(a)a^2 = a^3D(a) - 2aD(a)a + D(a)a^2
\]

\[= [a, [a, D(a)]] \subseteq Z,
\]

and so $D(a^{3+3}) = 3a^{3+2}D(a^3) = 0$. Hence $R$ is commutative by [3, Theorem 2].

**References**


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