Degrees of Self-maps

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DEGREES OF SELF-MAPS

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0. Introduction. As well-known, one of the basic problems in algebraic topology is to determine the homology representation

\[ H : [X, Y] \to \text{Hom}(H_*(X), H_*(Y)) \]

where \([X, Y]\) denotes the set of homotopy classes of maps from \(X\) to \(Y\).

In this paper we shall consider it for the case \(X = Y\). Previously, Sullivan and Quillen proved the famous theorem for \(X = \text{HP}^n\) in [7] and C. A. Mcgibbon determined the image of \(H\) for \(X = \text{RP}^n\), \(\text{CP}^n\) and \(\text{HP}^n\) in the stable case in [3]. Furthermore D. M. Davis investigated it for \(X = \text{CP}^{n+2}/\text{CP}^{n-1}\), the stunted projective space in [1] and also S. Sasao and M. Nagaishi determined it for \(X = \text{HP}^3\) in [6]. In this paper we shall consider the case \(X = S^n \cup e^{n+2} \cup e^{n+4}\), which is a generalization of Davis's case and contains the following:

1. The total spaces of \(S^2\)-bundles over \(S^4\).
2. The Thom complexes of real \(n\)-vector bundles over a 2-cell complex \(S^2 \cup e^4\), which contain the stunted complex projective spaces \(\text{CP}^{n+2}/\text{CP}^{n-1}\).
3. The iterated suspension of (1) or (2).

Let \(X\) be a 3-cell complex of the form \(S^n \cup e^{n+2} \cup e^{n+4}\), and \(e_j\) be the corresponding generator of \(H_j(X) \cong \mathbb{Z}\) for \(j = n, n+2, n+4\). Then for each self-map \(f \in [X, X]\), the endomorphism

\[ H(f) = f_* : H_*(X) \to H_*(X) \]

is uniquely determined by a triple of degrees \((d_1, d_2, d_3)\) which is defined by

\[ f_*(e^{(n-2)+2j}) = d_je^j \]

for \(j = 1, 2, 3\). We call \(f\) a self-map of \(X\) of degrees \((d_1, d_2, d_3)\).

Hence our problem is reduced to characterize a triple of integers \((d_1, d_2, d_3)\) which is a triple of degrees of a self-map of \(X\). This note is organized as follows:

In §1, we shall consider the case \(n = 2\), and investigate the case \(n = 3, 4\) in §2. In §3, we shall treat the case \(n > 4\) which is belonging to the stable range, and some examples shall be given in §4.

Remark. In a subsequent paper, we shall consider the kernel of \(H\).

Here we state a part of our results.

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Theorem A. For $X = S^1 \cup e^4 \cup e^8$, there exists a self-map of $X$ of degrees $(d_1, d_2, d_3)$ if and only if the followings hold:

1. If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) = 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3 \mod h_n(X)$.
2. If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) = 0$, $e^2 \cdot e^4 \neq 0$, then $d_2 = d_1^2$ and $d_3 \equiv d_1 \mod 2$.
3. If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) = 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_1 \mod h_n(X)$.
4. If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 = 0$, $Sq^2(e^4) = 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_2 \mod 2$.
5. If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) = 0$, then $d_3 \equiv d_3 \mod 2$.

Here $\cdot$ denotes the cup product in the cohomology ring and $Sq$ is the Steenrod squaring operation, and $h_n(X)$ denotes the image of the Hurewicz homomorphism at dimension $n$.

Corollary E. For $X = S^n \cup e^{n+2} \cup e^{n+4}$ ($5 \leq n$), a triple of integers $(d_1, d_2, d_3)$ is realizable by a self-map of $X$ if and only if the followings hold:

1. $d_3 \equiv d_1 \mod 2$ and $2d_3 \equiv 2d_1 \mod h_{n+4}(X)$ if $Sq^2(e^{n+2}) \neq 0$.
2. $d_3 \equiv d_1 \mod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^4(e^n) = 0$.
3. $d_3 \equiv d_1 \mod 2$ and $d_3 \equiv d_1 \mod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^4(e^n) \neq 0$.

The author wishes to thank the referee for his profound criticism.

1. The case $n = 2$. Let $X$ be a 3-cell complex of the form $S^2 \cup e^4 \cup e^8$. Let $a$ and $b$ be integers and $\varepsilon \in \{0, 1\}$. Then we call $X$ to be of type $(a, b, \varepsilon)$ if and only if

$$e^2 \cdot e^2 = ae^4, \quad e^2 \cdot e^4 = be^6, \quad \text{and} \quad Sq^2(e^4) = \varepsilon e^6$$

where $e^i$ denotes the generator of $H^i(X, \mathbb{Z})$ (or $H^i(X, \mathbb{Z}/2)$). From now on in this section, we assume that $X$ is of type $(a, b, \varepsilon)$.

First we consider the sub-case

1.1 $a = 0$ (i.e. $e^2 \cdot e^2 = 0$) By the assumption we may consider that $X$ has a form $X = (S^2 \vee S^4) \cup e^8$. Then, the attaching class $\beta$ for the cell $e^8$ is the following

$$\beta = x_{\eta_2} (\eta_2 \eta_3 \eta_4) + \omega (\eta_4) + b [\omega_2, \omega_4]$$

where $\eta_n$ denotes the Hopf class of $\pi_{n-1}(S^n)$.

Now define two maps $S^2 \vee S^4 \to S^2 \vee S^4$ as follows:
\[ \phi_{k,\ell}|S^2 = k\ell \quad \text{and} \quad \phi_{k,\ell}|S^4 = \ell \eta_3. \]
\[ \phi_{k,\ell}|S^2 = k\ell \quad \text{and} \quad \phi_{k,\ell}|S^4 = \ell \eta_4 + \ell_2 \eta_2 \eta_3. \]

**Lemma 1.** When \( b \neq 0 \), there exists a self-map of \( X \) of degrees \((k, \ell, m)\) if and only if \( m = k\ell \) and \( m \equiv \ell \mod 2 \).

**Proof.** By the standard argument we can easily know that there is a self-map of \( X \) of degrees \((k, \ell, m)\) if and only if \( \phi_{k,\ell}(\beta) = m\beta \) or \( \phi_{k,\ell}(\beta) = m\beta \). Since two endomorphisms

\[ \psi_{k,\ell} \text{ and } \phi_{k,\ell} : \pi_3(S^2 \vee S^4) \to \pi_3(S^2 \vee S^4) \]

are clearly given by

\[ \phi_{k,\ell}(\beta) = k\ell \eta_2 \eta_3 \eta_4 + \ell_2 \eta_2 \eta_3 \eta_4 + \ell \eta_3 \eta_4 + k\ell b[\ell_2, \eta_3], \]
\[ \phi_{k,\ell}(\beta) = k\ell \eta_2 \eta_3 \eta_4 + \ell \eta_3 \eta_4 + k\ell b[\ell_2, \eta_3] \]  

(1-1),

the condition is equivalent to \( mb = k\ell b \), \( \ell \equiv m \mod 2 \), and \( k\ell \eta_2 \eta_3 \eta_4 = m\ell \eta_2 \eta_3 \eta_4 \mod b \eta_2 \eta_3 \eta_4 \). Then the assumption \( b \neq 0 \) completes the proof.

Now assume that \( b = 0 \) and \( c \neq 0 \). Then, from (1-1) we can obtain the following:

**Lemma 2.** When \( \varepsilon = 0 \) and \( b \neq 0 \), there is a self-map of \( X \) of degrees \((k, \ell, m)\) if and only if \( m = k\ell \) and \( m \equiv k \mod h_\ell(\Sigma X) \), where \( \Sigma \) denotes the suspension functor.

Analogously we have

**Lemma 3.** When \( b = 0 \), there exists a self-map of \( X \) of degrees \((k, \ell, m)\) if and only if

1. \( m \equiv k \mod h_\ell(X) \) if \( \varepsilon = 0 \).
2. \( m \equiv \ell \mod 2 \) if \( \varepsilon \neq 0 \).

1.2 \( a \neq 0 \) (i.e. \( e^2 \cdot e^3 \neq 0 \)). Let \( A_a \) be the 2 cell-complex \( S^2 \cup e^4 \) which has a \( a\eta_2 \) as the attaching class for the cell \( e^4 \). First we quote the following ((2.13) of [8]):
Lemma 4.

\[ \pi_0(A_a) = \begin{cases} 
\mathbb{Z} & a \equiv 1 \pmod{2} \\
\mathbb{Z} + \mathbb{Z}/4 & a \equiv 2 \pmod{4} \\
\mathbb{Z} + \mathbb{Z}/2 + \mathbb{Z}/2 & a \equiv 0 \pmod{4}
\end{cases} \]

Let \( \phi_a \colon A_a \rightarrow A_a \) be a map of degrees \((k, k^3)\), then other self-maps of \( A_a \) of degrees \((k, k^3)\) is only one, which is given by the composite

\[ \psi_a \colon c \rightarrow A_a \vee S^4 \rightarrow A_a = \psi_a \vee \tau_2 \eta_2 \eta_3 \cdot c \]

Let \( \beta \) be the attaching class for the cell \( e^6 \) of \( X \) and \( \beta' \) be the image of \( \beta \) by the pinching map \( A_a \rightarrow S^4 = A_a / S^2 \). Clearly \( \beta' \) is 0 or \( \eta_3 \), which is determined by \( Sq^2(e^4) \). Using the fact \( [\tau_2, \eta_2 \eta_3] = 0 \), we can easily obtain the following:

\[ \phi_a \cdot (\beta) = k^3 \beta + \tau_2 \gamma \quad \text{and} \quad \psi_a \cdot (\beta) = \psi_a \cdot (\beta) + \tau_2 \eta_2 \eta_3 \beta' \quad (1-2) \]

where \( \gamma \) is an element of \( \pi_5(S^2) \), i.e., 0 or \( \eta_2 \eta_3 \eta_4 \).

Lemma 5. If \( a \equiv 1 \pmod{2} \), then we have \( \phi_a \cdot (\beta) = \psi_a \cdot (\beta) = k^3 \beta \).

Proof. Lemma 4 implies that \( \gamma \) and \( \beta' \) in the formula (1-2) is always 0. Hence the proof is complete.

Next we investigate the case \( a \equiv 0 \pmod{2} \) (\( a \neq 0 \)). First we prove

Lemma 6. For \( a = 2 \), we can choose \( \psi_a \) satisfying \( \psi_a \cdot (\beta) = k^3 \beta \).

Proof. We may regard \( A_2 \) as the 4-skelton of the reduced product \( S^2 \) of \( S^2 \) in [2]. The map \( k\tau_2 \colon S^2 \rightarrow S^2 \) induces the map \( S^2 \rightarrow S^2 \) whose restriction on its 4-skelton is the desired map \( \psi_a \), and \( \psi_a \cdot (a) = k^3 a \) holds for \( a \), the attaching class for the 6-cell of \( S^2 \). On the other hand, from lemma 4 and the homotopy exact sequence of the pair \((A_a, S^4)\) we can know that there is a class \( a_i \) of \( \pi_0(A_a) \) satisfying \( j_*(a_i) = [\chi_i, \tau_2] \), where \([, \) denotes the relative Whitehead product and \( \chi_i \) is the characteristic map for the cell \( e^4 \) of \( S^2 \). Since we have \( e^2 \cdot e^4 = 3e^6 \) in \( H(S^2) \) (see [2]), the following relations hold

\[ 3a_1 = a \quad \text{or} \quad 3a_i = a + \eta_2 \eta_3 \eta_4. \]

Then we have

\[ 3\psi_a \cdot (a) = \phi_a \cdot (a) = k^3 a = 3k^3 a_1 \quad \text{or} \]
\[ 3\psi_a \cdot (a) = \phi_a \cdot (a) + k\eta_2 \eta_3 \eta_4 = k^3 a + k\eta_2 \eta_3 \eta_4 = k^3 (a + \eta_2 \eta_3 \eta_4) = 3k^3 a_1 \quad \text{i.e.} \]
\( \psi_k(\alpha) = k^3 \alpha. \)

Hence, for \( \beta'(\in \pi_6(A_2)) \) with \( \beta' = 0 \), it holds \( \psi_{\phi}(\beta) = k^3 \beta \). Moreover, if \( \beta' = 0 \), we obtain from the formula (1-2) that \( \psi_{\phi}(\beta) = k^3 \beta \) or \( \psi_{\phi}(\beta) = k^3 \beta \). Thus the proof is complete.

Secondly, we prove the general case.

**Lemma 7.** If \( a \equiv 0 \mod 2 \), then there is a map \( \phi_k : A_2 \to A_2 \) satisfying \( \phi_{\phi_\beta}(\beta) = k^3 \beta \) for any \( \beta \) of \( \pi_6(A_2) \).

**Proof.** For \( \beta \) with \( \beta' \neq 0 \), the proof follows from the formula (1-2). If \( \beta' = 0 \) we put \( a = 2a' \). Since, for any map \( \phi(1, a') : A_2 \to A_2 \) of degrees \( (1, a') \), there exists a map \( \phi_k : A_2 \to A_2 \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\phi_{(1, a')}} & A_2 \\
\downarrow \phi_k & & \downarrow \phi_k \\
A_2 & \xrightarrow{\phi(1, a')} & A_2
\end{array}
\]

Then, the proof follows from the formula (1-1), lemma 6, and the restriction \( \phi(1, a')|S^2 = \text{the identity} \).

**Proposition 1.** For \( a \neq 0 \), a triple \( (d_1, d_2, d_3) \) is realizable by a self-map of \( X \) if and only if the followings hold:

1. \( d_2 = k^2 \).
2. If \( \varepsilon = 0 \), then \( d_3 \equiv d_1^2 \mod h_6(X) \).
3. If \( \varepsilon \neq 0 \) and \( b \neq 0 \), then \( d_3 = d_1^2 \).
4. If \( \varepsilon \neq 0 \) and \( b = 0 \), then \( d_3 \equiv d_1^2 \mod 2 \).

**Proof.** (0) is easy by the assumption \( a \neq 0 \). Using lemma 5 and 7, we may take \( \gamma = 0 \) in the formula (1-1). Then our desired condition is

\[ m\beta = k^3 \beta \quad \text{or} \quad m\beta = k^3 \beta + \nu_2 \eta_2 \eta_3 \beta'. \]

Thus the proof of (1), (2), and (3) follows from using that \( \beta' \) is determined by \( Sq^7(e') \).

Now we prove Theorem A. Namely, (3) follows from lemma 2. (4) follows from lemma 1. (5) and (6) follow from lemma 3. The others follow from lemma 5 and proposition 1.

**2. The case \( n = 3, 4 \).** The case can be divided into two ones by the formula
Mathematical Journal of Okayama University, Vol. 34 [1992], Iss. 1, Art. 21

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\[ Sq^2 Sq^2 = Sq^3 Sq^1: \]

(1) \[ Sq^3(e^{n+2}) \neq 0, \] then \[ X = (S^n \vee S^{n+2}) \cup e^{n+4} \] because of \( Sq^2(e^n) = 0. \)

(2) \[ Sq^3(e^{n+2}) = 0, \] then \[ X = e^{n+4} \cup S^n \cup e^{n+2}. \]

Let \( \beta \) be the attaching class for the \((n+4)\)-cell of \( X \). First we consider the case (1). Define two maps \( \psi_{k,t} \) and \( \psi'_{k,t} \) as follows:

\[ \psi_{k,t}, \psi'_{k,t}: S^n \vee S^{n+2} \rightarrow S^n \vee S^{n+2}, \]
\[ \psi_{k,t}|S^n = kt_n \quad \text{and} \quad \psi_{k,t}|S^{n+2} = \ell t_{n+2}, \]
\[ \psi'_{k,t}|S^n = kt_n \quad \text{and} \quad \psi'_{k,t}|S^{n+2} = \ell t_{n+2} + \iota_n \eta_n \eta_{n+1}. \]

Then we have

\[ \psi_{k,t} (\beta) = (kt_n) \ast (\beta_1) + \iota_n (\ell \eta_{n+2}), \]

and

\[ \psi'_{k,t} (\beta) = (kt_n) \ast (\beta_1) + \iota_n \eta_n \eta_{n+1} \eta_{n+2} + \iota_n (\ell \eta_{n+2}), \]

where \( \beta = \beta_1 + \iota_n \eta_{n+2} \) \((\in \pi_{n+3}(S^n \vee S^{n+2}) = \pi_{n+3}(S^n) + \pi_{n+3}(S^{n+2}))\). Hence, a triple \((k, \ell, m)\) is realizable by a self-map of \( X \) if and only if the following equality holds:

\[ m \beta_1 + m \iota_{n+2} \eta_{n+2} = m \beta = (kt_n) \ast \beta_1 + \ell \iota_{n+2} \eta_{n+2} \mod \iota_n \eta_n \eta_{n+1} \eta_{n+2}. \quad (2.1) \]

**Lemma 8.** For \( n = 3 \) \((Sq^3(e^0) \neq 0)\), there exists a self-map of \( X \) of degrees \((k, \ell, m)\) if and only if \( \ell = m \mod 2 \) and \( 2m = 2k \mod h_1(X) \).

**Proof.** Since we can replace \((kt_n) \ast (\beta_1)\) with \(k \beta_1\) in the formula (2.1) we have that \( \ell = m \mod 2 \) and \((m-k) \beta_1 = 0 \mod \eta_n \eta_{n+2} \). On the other hand, \( \eta_n \eta_{n+2} \) is the only one element of order 2 in \( \pi_6(S^3) \). Therefore the latter condition is equivalent to \( 2(m-k) \beta_1 = 0 \). Thus the proof is completed by \( 2 \beta_1 = 2 \beta. \)

**Lemma 9.** For \( n = 4 \) \((Sq^3(e^0) \neq 0)\), our conditions are as follows:

(1) If \( e^4 \ast e^4 = 0 \), then \( \ell = m \mod 2 \) and \( 2(m-k) \equiv 0 \mod h_6(X) \).

(2) If \( e^4 \ast e^4 \neq 0 \), then \( m = k^2 \), \( \ell = m \mod 2 \), and \( m = k \mod h_6(SX) \).

**Proof.** Since \( \beta_1 \) has a representation

\[ \beta_1 = x \nu + y \Sigma \omega \quad \text{for some integers} \ x \text{ and} \ y, \]

where \( \nu \) denotes the Hopf map \( S^7 \rightarrow S^4 \) and \( \omega \) is the Blaker-Massey map, we get

\[ (kt_n) \ast (\beta_1) = k \beta_1 + k(k-1)/2[I_n, \omega]H(\beta_1) \]
\[ = k^2 x \nu + [ky + k(k-1)x/2] \Sigma \omega. \]

Hence, the formula (2-1) is equivalent to
\[ mx + my \Sigma \omega = k^2 x + [(2ky + k(k-1)x)/2] \Sigma \omega \mod \eta_4 \eta_5 \eta_6. \]

Moreover this gives that

if \( x \neq 0 \) (i.e. \( e^4 \cdot e^4 \neq 0 \)), then \( m = k^2 \) and \( (2my - 2ky - k(k-1)x)/2 \Sigma \omega \equiv 0 \mod \eta_4 \eta_5 \eta_6 \), and that

if \( x = 0 \) (i.e. \( e^4 \cdot e^4 = 0 \)), then \( (m - k) y \Sigma \omega \equiv 0 \mod \eta_4 \eta_5 \eta_6 \).

Thus (1) is obtained from the same argument as lemma 8. Next, we consider the case (2). From \( m = k^2 \) we have that

\[ 2my - 2ky - k(k-1)x = (m - k) 2y - k(k-1)x = k(k-1)(2y - x). \]

On the other hand, we know that \( \Sigma \beta_1 = (x - 2y) \nu \). Thus the proof of (2) follows from \( k(k-1) \Sigma \beta_1 = k(k-1) \Sigma \beta \).

Secondly, we prove the case \( Sq^2(e^{n+2}) = 0 \). Let \( A \) be the subcomplex, \( S^n \cup e^{n+2} \), of \( X \) and let \( \phi_\gamma(\beta) \) be the map defined by

\[ \phi_\gamma(\gamma) = (k1_\beta \vee \gamma) C_\beta : A \to A \vee S^{n+2} \to A \]

for \( \gamma \in \pi_{n+2}(A) \) where \( k1_\beta \) denotes the \( k \) time of the identity of \( A \) in the sense of the suspension-addition and \( C_\beta \) is the co-action map of \( A \). Here we note that

(0) \( \beta = i_\ast(\beta') \), where \( i \) is the inclusion \( S^n \to A \).

(1) \( \phi_\gamma(\gamma) \) is of degrees \( (k, k + h_{n+2}(X)) \).

(2) \( \phi_\gamma(\gamma) \ast (\beta) = i_\ast((k1_\beta) \ast (\beta')) \).

Now consider the following diagram:

\[ \begin{array}{c}
\pi_{n+3}(S^n) \xrightarrow{i_\ast} \pi_{n+3}(A) \\
\downarrow (k1_\beta) \ast \\
\pi_{n+4}(A, S^n) \xrightarrow{\delta} \pi_{n+3}(S^n) \xrightarrow{i_\ast} \pi_{n+3}(A)
\end{array} \]

Then, it is easy from \( \phi_\gamma(\gamma) \ast (\beta) = (k1_\beta) \ast (\beta) \) to obtain the following:

**Lemma 10.** There exists a self-map of \( X \) of degrees \( (k, \ell, m) \) if and only if \( \ell \equiv k \mod h_{n+2}(X) \) and \( m(\beta') \equiv (k1_\beta) \ast (\beta') \mod \partial \)-image.

**Remark.** \( h_{n+2}(X) = Z \) if \( Sq^2(e^n) = 0 \) and \( h_{n+2}(X) = 2Z \) if \( Sq^2(e^n) \neq 0 \).

Since \( (k1_\beta) \ast (\beta') = k \beta \) (for \( n = 3 \)), we have

**Lemma 11.** For \( n = 3 \), a triple \( (k, \ell, m) \) is realizable by a self-map of \( X \)
if and only if $\ell \equiv k \mod h_5(X)$ and $m \equiv k \mod h_7(X)$.

If $n = 4$ we can take $x \nu + y \Sigma \omega$ as $\beta'$ and use the formula,

\[
(k \nu)_*(\beta') = k(\beta') + \frac{k(k-1)}{2} x [\nu, \nu].
\]

(2-2)

Lemma 12. If $e^* \cdot e^* = 0$, then there is a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $\ell \equiv k \mod h_5(X)$ and $m \equiv k \mod h_8(X)$.

Proof. Since $e^* \cdot e^* = 0$ is equivalent to $x = 0$, we have

\[
(k \nu)_*(\beta') = k(\beta'), \quad \text{i.e.} \quad (k \nu)_*(\beta') = k \beta
\]

from (2-2). Hence $(m-k)(\beta) = 0$, which completes the proof.

Lemma 13. If $x \neq 0$, then there is a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $\ell \equiv k \mod h_8(X)$, $m = k^2$, and $m \equiv k \mod h_8(\Sigma X)$.

Proof. First, we have $m = k^2$ from (2-2) and $[\nu, \nu] = 2 \nu + \Sigma \omega$. Then, it holds $i_*[(m-k)(x-2y)/2] \Sigma \omega = 0$, which gives

\[
i_*[(m-k)(x-2y)v] = i_*[(m-k)\Sigma \beta'] = (m-k)\Sigma \beta = 0
\]

from applying the suspension functor. Thus the proof is complete.

Now from lemmas 8, 9, 11, 12, and 13 we have

Theorem B. Let $X$ be a complex of the form $S^5 \cup e^5 \cup e^7$. Then a triple $(d_1, d_2, d_3)$ is realizable by a self-map of $X$ if and only if

1. If $Sq^4(e^5) \neq 0$, then $d_3 \equiv d_2 \mod 2$ and $2d_3 \equiv 2d_2 \mod h_7(X)$.
2. If $Sq^4(e^5) = 0$ and $Sq^4(e^3) = 0$, then $d_3 \equiv d_1 \mod h_7(X)$.
3. If $Sq^4(e^3) = 0$ and $Sq^4(e^3) \neq 0$, then $d_3 \equiv d_1 \mod h_7(X)$ and $d_2 \equiv d_1 \mod 2$.

Theorem C. Let $X$ be a complex of the form $S^4 \cup e^6 \cup e^8$. Then a triple $(d_1, d_2, d_3)$ is realizable by a self-map of $X$ if and only if

1. If $Sq^4(e^6) \neq 0$ and $e^4 \cdot e^4 = 0$, then $d_3 \equiv d_2 \mod 2$ and $2d_3 \equiv 2d_1 \mod h_8(X)$.
2. If $Sq^4(e^6) = 0$ and $e^4 \cdot e^4 \neq 0$, then $d_3 \equiv d_2 \mod 2$, $d_3 \equiv d_1^2$, and $d_3 \equiv d_2 \mod h_8(\Sigma X)$.
3. If $Sq^4(e^6) = 0$ and $e^4 \cdot e^4 = 0$, then $d_2 \equiv d_1 \mod h_8(X)$ and $d_3 \equiv d_1 \mod h_8(X)$.
(4) If $Sq^2(e^0) = 0$ and $e^4 \cdot e^4 \neq 0$, then $d_2 \equiv d_1 \mod h_6(X)$, $d_3 = d_2^2$, and $d_3 \equiv d_1 \mod h_9(\Sigma X)$.

3. The case $X = \Sigma X'$. First we note that this case contains the case $5 \leq n$. Next, let $Y$ be the complex $X/S^n$ and let us consider maps from $Y$ to $X$.

**Lemma 14.** If $Sq^2(e^{n+2}) = 0$ in $\text{H}^*(X; \mathbb{Z}/2)$, then there exists a map from $Y$ to $X$ with degrees $(\ell, m)$ if and only if $\ell \equiv 0 \mod h_{n+2}(X)$ and $m \equiv 0 \mod h_{n+4}(X)$.

**Proof.** Since the assumption implies $Y = S^{n+2} \vee S^{n+4}$, the proof is clear.

**Lemma 15.** If $Sq^2(e^{n+2}) \neq 0$, then there is a map from $Y$ to $X$ with degrees $(\ell, m)$ if and only if $m \equiv \ell \mod 2$ and $2m \equiv 0 \mod h_{n+4}(X)$.

**Proof.** Since $Sq^2(e^{n+2}) \neq 0$ implies $Sq^2(e^n) = 0$, $X$ has a decomposition $X = (S^n \vee S^{n+2}) \cup e^{n+4}$.

Let $f : S^{n+2} \to X$ be the map defined by $f = n\gamma + \ell\eta_{n+2}$ for $\gamma \in \pi_{n+2}(S^n)$ and $\beta = c_n\beta' + c_{n+2}\eta_{n+2}$ be the attaching class for the $(n+4)$-cell of $X$. Then, we have

$$f_\ast(\eta_{n+2}) = \ell_\ast(\eta_{n+2}) + c_{n+2}\eta_{n+2}.$$

Hence the proof follows from $m\beta \equiv m_c\beta' + mc_{n+2}\eta_{n+2} = f_\ast(\eta_{n+2})$, i.e. $m \equiv \ell \mod 2$ and $m\beta' \equiv 0 \mod c_{n+2}\eta_{n+2}$.

Secondly, using the group structure of $[X, X] = [\Sigma X', \Sigma X']$ we have the map $k1_X$, which is of degrees $(k, k, k)$. Let $g : X \to X$ be a self-map of degrees $(k, \ell, m)$. Then the map $g - k1_X$ is of degrees $(0, \ell - k, m - k)$, and lemmas 14 and 15 give the following

**Theorem D.** Assume that $X$ is a suspended space. Then a triple $(d_1, d_2, d_3)$ is realizable by a self-map of $X$ if and only if

1. If $Sq^2(e^{n+2}) = 0$, then $d_2 \equiv d_1 \mod h_{n+2}(X)$ and $d_3 \equiv d_1 \mod h_{n+4}(X)$.
2. If $Sq^2(e^{n+2}) \neq 0$, then $d_2 \equiv d_1 \mod 2$ and $2d_3 \equiv 2d_1 \mod h_{n+4}(X)$.

**Remark.** If $Sq^2(e^n) = 0$, then $h_{n+2}(X) = \mathbb{Z}$, and if $Sq^2(e^n) \neq 0$, then $h_{n+2}(X) = 2\mathbb{Z}$. 

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4. Examples.

Example 1. Let $X_n$ be the space $\mathbb{CP}^{n-2}/\mathbb{CP}^{n-1}$ ($1 \leq n$). Then, using the results in [5] and $Sq^2(e^r) = ie^{r+1}$ in $H^*(\mathbb{CP}^n)$, we have that degrees of self-maps of $X_n$, $(d_1, d_2, d_3)$, is characterized as follows:

1. For $3 \leq n$,
   - if $n \equiv 1 \mod 2$, then $d_2 \equiv d_1 \mod 2$ and $d_3 \equiv d_1 \mod 24/(n+3, 24)$,
   - if $n \equiv 0 \mod 8$, then $d_2 \equiv d_3 \mod 2$ and $2d_5 \equiv 2d_1 \mod 96/(n, 48)$,
   - if $n \equiv 2, 4, 6 \mod 8$, then $d_2 \equiv d_3 \mod 2$ and $2d_5 \equiv 2d_1 \mod 48/(n, 48)$,

where $(,)$ denotes the greatest common divisor of integers.

Remark. These results also hold for the iterated-suspension $\Sigma^s X_n$.

2. For $n = 2$, $d_3 \equiv d_2 \mod 2$ and $d_2 \equiv d_1$, i.e. $(k, k^2+2Z, k^3)$, and for the space $\Sigma^s X_2$ ($1 \leq s$) we have $(d_1, d_2, d_3) = (k, k^2 Z, k+12Z)$.

3. For $n = 1$, $(d_1, d_2, d_3) = (k, k^2, k^3)$, and for $\Sigma^s X_1$ ($1 \leq s$) we have $(d_1, d_2, d_3) = (k, k+2Z, k+6Z)$.

Example 2. Let $X$ be the 6-skelton of the reduced product of $S^2$. Since it is clear that $\Sigma^s X$ is decomposed into $S^{2+s} \cup S^{4+s} \cup S^{6+s}$ ($1 \leq s$) we can know that

1. For $s = 0$, $(d_1, d_2, d_3) = (k, k^2, k^3)$,
2. For $1 \leq s$, $(d_1, d_2, d_3)$ for any $d_1, d_2$ and $d_3$.

Example 3. Let $X_r$ be the 2-sphere bundle over $S^4$ whose characteristic class is $r$ times of a generator ($\in \pi_3(SO(3)) = Z$). Then it is easy to see

$$e^2 \cdot e^2 = re^4 \quad \text{and} \quad e^2 \cdot e^4 = e^6.$$ 

Furthermore the suspension $\Sigma^N X_r$ has a decomposition

$$\Sigma^N X_r = S^{N+2} \cup e^{N+4} \cup e^{N+6}$$

in which the attaching class $\beta_N$ for the $(N+6)$-cell is given by

$$\beta_N = i_*(\Sigma^{N-1}J(\xi_r)) = i_*(r\Sigma^{N-1} \omega) = ri_*(\Sigma^{N-1} \omega)$$

where $\xi_r$ is the real vector bundle associated with $X_r$.

On the other hand, it is easy to show that

$Sq^2(e^{N+4}) = 0$ ($0 \leq N$) and $Sq^2(e^{N+3}) = 0$ for even $r$, $\neq 0$ for odd $r$ ($1 \leq N$).

Since $Sq^2(e^{N+4}) = 0$ implies that $i^*$ is injective, we get

$$h_{N+6}(\Sigma^N X_r) = (r, 12Z).$$
These facts give the following:

A triple $(d_1, d_2, d_3)$ is realizable by a self-map of $\Sigma^nX_r$ if and only if

1. The case $1 \leq N$.
   
   $d_3 \equiv d_1 \mod(r, 12)$ if $r \equiv 0 \mod 2$.
   
   $d_2 \equiv d_1 \mod 2$ and $d_3 \equiv d_1 \mod(r, 12)$ if $r \equiv 1 \mod 2$.

2. The case $N = 0$.
   
   $d_3 = d_1d_2$ if $r = 0$.
   
   $d_2 = d_1^2$ and $d_3 = d_1^3$ if $r \neq 0$.

**Example 4.** Let $Y_a$ be the CW-complex $S^4 \cup e^4$ which has $a\eta_2$ as the attaching class for the cell $e^4$. Assume $5 \leq N$ and consider an $N$-dim real vector bundle $\xi$ over $Y_a$. For simplicity we suppose that its Stiefel-Whitney class $w_2(\xi)$ is trivial. Since $w_2(\xi) = 0$ implies that the restriction $\xi|S^2$ is trivial, there is a commutative diagram

$$
\xi \rightarrow \xi' \\
\downarrow \quad \quad \downarrow \\
Y_a \rightarrow S^4 = Y_a/S^2
$$

for some $\xi'$.

Then, from $Sq^2(e^{n+2}) = ae^{n+4}$ ([4]), we can know that the Thom complex $T(\xi)$ has a decomposition

$$
T(\xi) = (S^n \vee S^{n+2}) \cup e^{n+4}
$$

in which $\beta = \omega J(\xi') + \omega_{n+2}(a\eta_{n+2}) \in \pi_{n+3}(S^n) + \pi_{n+3}(S^{n+2})$. Thus, from Theorem D, we have that a triple $(d_1, d_2, d_3)$ is realizable by a self-map of $T(\xi)$ if and only if

1. if $a \equiv 0 \mod 2$, then $d_3 \equiv d_1 \mod(b, 24)$,
2. if $a \equiv 1 \mod 2$, then $d_3 \equiv d_1 \mod 2$ and $2d_3 = 2d_1 \mod(b, 12)$,

where $2be^4 = p_1(\xi)$, the first Pontrjagin class of $\xi$.

**References**


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(Received June 19, 1992)