On H-separable Polynomials in Skew Polynomial Rings of Automorphism Type

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ON H-SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS OF AUTOMORPHISM TYPE

Dedicated to Professor Manabu Harada on his 60th birthday

Shōichi IKEHATA and George SZETO

In [2], [3] and [4], one of the authors has studied $H$-separable polynomials in skew polynomial rings. In [4], we have studied $H$-separable polynomials of prime degree in skew polynomial rings of automorphism type. The present paper is a natural continuation of [4].

Throughout this paper, $B$ will represent a ring with 1, and $\rho$ an automorphism of $B$. Let $B[X; \rho]$ be the skew polynomial ring in which the multiplication is given by $bX = X\rho(b)$ ($b \in B$). A ring extension $S/B$ is called a separable extension if the $S$-$S$-homomorphism of $S \otimes_S S$ onto $S$ defined by $a \otimes b \mapsto ab$ splits, and $S/B$ is called an $H$-separable extension if $S \otimes_S S$ is $S$-isomorphic to a direct summand of a finite direct sum of copies of $S$. A monic polynomial $f$ in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]/f$ is called a separable (resp. $H$-separable) polynomial if $B[X; \rho]/fB[X; \rho]$ is a separable (resp. $H$-separable) extension of $B$. It is well known that every $H$-separable extension is a separable extension. As to terminologies used in this note, we follow [2].

In [4, Theorem 2], for any prime number $p$, we have shown that the center $Z$ of $B$ is a Galois extension over $Z^*$ with the Galois group $(\rho|Z)$ whose order is $p$ if and only if $B[X; \rho]$ contains an $H$-separable polynomial of degree $p$. In this paper, for general $m$, we shall characterize the condition that $Z$ is a Galois extension over $Z^*$ with the Galois group $(\rho|Z)$ whose order is $m$ in terms of $H$-separable extensions (Theorem 1). Moreover, we shall obtain a sharpening of [4, Theorem 4]. Some more results will be obtained in [5].

We shall use the following conventions:

$Z = \text{the center of } B$.

$V_S(B) = \text{the centralizer of } B \text{ in } S \text{ for a ring extension } S/B$.

$B^* = \{a \in B \mid \rho(a) = a\}$, $Z^* = \{a \in Z \mid \rho(a) = a\}$.

Let $f$ be a monic polynomial in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]/f$. Then we shall denote $B[x; \rho] = B[X; \rho]/fB[X; \rho]$, where $x = X + B[X; \rho]/fB[X; \rho]$, and $B[x^t; \rho^t] = \text{the subring of } B[x; \rho] \text{ generated by } B$ and $x^t$.

Recall that if $f$ is an $H$-separable polynomial in $B[X; \rho]$ of degree $m$, then $f = X^m - u$, $u$ is invertible in $B^*$ and $au = u\rho^m(a)$ ($a \in B$) ([3, Lemma 1]).
First, we shall state the following theorem which is a generalization of [4, Theorem 2].

**Theorem 1.** Let \( f = X^n - u \) be in \( B[X; \rho] \) with \( fB[X; \rho] = B[X; \rho]f \). Then the following are equivalent:

(a) \( u \) is invertible in \( B^\rho \), and \( Z/Z^\rho \) is a \( G \)-Galois extension, where \( G \) is the group generated by \( \rho|Z \) of order \( m \).

(b) \( B[x^n; \rho^n] \) is an \( H \)-separable extension over \( B \) for every divisor \( n \) of \( m \).

**Proof:** (a) \( \implies \) (b). Assume \( m = nd \). Then we have

\[
B[x^n; \rho^n] \cong B[Y; \rho^n]/(Y^d - u)B[Y; \rho^n],
\]

where \( \alpha Y = Y\rho^n(a) (\alpha \in B) \), and it is clear that \( (Y^d - u)B[Y; \rho] = B[Y; \rho](Y^d - u) \). Since \( Z/Z^\rho \) is a \( G \)-Galois extension, \( Z/Z^n \) is a \( \rho^n|Z \)-Galois extension and \( \rho^n|Z \) is of order \( d \). Hence, by [2, Proposition 1.4] \( Y^d - u \) is an \( H \)-separable polynomial in \( B[Y; \rho^n] \), and so \( B[x^n; \rho^n] \) is an \( H \)-separable extension \( B \).

To prove the converse, we need the following elementary lemma:

**Lemma 2.** If there exist divisors \( n_1, n_2, \ldots, n_r \) of \( m \) such that \( m = n_1n_2 \cdots n_r \) and in the tower

\[
Z = Z^\rho \supset Z^\rho_{n_1} \supset Z^\rho_{n_2} \cdots \supset Z^\rho_{n_r} \supset Z^\rho,
\]

each \( Z^\rho_{n_1} \supset Z^\rho_{n_2} \sqcup \cdots \sqcup Z^\rho_{n_r} \supset Z^\rho \) is a \( (\rho|Z^{n_1} \supset Z^{n_2} \cdots \supset Z^{n_r}) \)-Galois extension, where the group \( (\rho|Z^{n_1} \supset Z^{n_2} \cdots \supset Z^{n_r}) \) is of order \( n_i \) (\( 1 \leq i \leq r \)), then \( Z/Z^\rho \) is a \( (\rho|Z) \)-Galois extension, where the group \( (\rho|Z) \) is of order \( m \).

**Proof.** By [1, Theorem 1.3], there exist elements \( a_k^{(i)}, b_k^{(i)} \in Z^\rho_{n_1} \cdots Z^\rho_{n_i} \) such that

\[
\sum_k a_k^{(i)}(\rho^{n_1-1} \cdots \rho^{n_i-1})(b_k^{(i)})^r = \delta_{0, \nu} (0 \leq \nu \leq n_i - 1, 1 \leq i \leq r).
\]

Then we can easily verify that

\[
\sum_{k_1, k_2, \ldots, k_r} a_{k_1}^{(i)} a_{k_2}^{(j)} \cdots a_{k_r}^{(r)} \rho^r b_{k_1}^{(i)} b_{k_2}^{(j-1)} \cdots b_{k_r}^{(r)} = \delta_{0, \nu} (0 \leq \nu \leq m - 1).
\]

Therefore we have the assertion by [1, Theorem 1.3] again.

Now, we come back to prove Theorem 1. (b) \( \implies \) (a).

Assume \( m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \), where each \( p_i \) are different prime numbers and \( e_i > 0 \). We define the sequence \( n_1, n_2, \ldots, n_r \) of divisors of \( m \) as follows:
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\[ n_i = \begin{cases} 
  p_i & (1 \leq i \leq e_1) \\
  p_i & (1 + e_1 \leq i \leq e_1 + e_2) \\
  \cdots \\
  p_h & (1 + e_1 + e_2 + \cdots e_{h-1} \leq i \leq e_1 + e_2 + \cdots + e_h) \\
  \end{cases} \]

\[ r = e_1 + e_2 + \cdots + e_h \quad \text{and} \quad 1 \leq i \leq r. \]

Then \( m = n_1 n_2 \cdots n_r \). We shall prove that in the tower

\[ Z = Z^{p_m} \supset \cdots \supset Z^{p_{n_1+n_2+\cdots+n_r}} \supset Z^{p_{n_1+n_2+\cdots+n_r}} \supset \cdots \supset Z^{p_{n_1+n_2+\cdots+n_r}} \supset Z^*, \]

each \( Z^{p_{n_1+n_2+\cdots+n_r}}/Z^{p_{n_i+1+n_2+\cdots+n_r}} \) is a \((\rho^{n_1+n_2+\cdots+n_r})\)-Galois extension of order \( n_i \) \((1 \leq i \leq r)\).

We put here \( s = n_{i+1} n_{i+2} \cdots n_r \) and \( t = n_{i+1} n_{i+2} \cdots n_r \). Then \( t = s n_i \), and we may assume

\[ t = p_j^{r_{j+1}} p_{j+1}^{r_{j+1}} \cdots p_k^{r_k} \quad \text{and} \quad s = p_j^{r_j} p_{j+1}^{r_{j+1}} \cdots p_k^{r_k}, \quad \text{so} \quad t = s p_j. \]

Now we have

\[ B[x^s; \rho^s] \cong B[x^s; \rho^t] = B[x^{sp_j}; \rho^{sp_j}] \supset B. \]

Since \( B[x^s; \rho^s] \cong B[Y; \rho^t]/(Y^q - u)B[Y; \rho^t] \), where \( m = qt \), \( Y^q - u \) is an \( H \)-separable polynomial in \( B[Y; \rho^t] \). Naturally, we can extend \( \rho^s \) to the automorphism \( \tilde{\rho}^s \) of \( B[x^{sp_j}; \rho^{sp_j}] \). Consider the skew polynomial ring \( B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s] \), where \( \alpha T = T\tilde{\rho}^s(\alpha) \) \((\alpha \in B[x^{sp_j}; \rho^{sp_j}]\) \). Then we have the following \( B[x^{sp_j}; \rho^{sp_j}]-\text{ring isomorphism} \)

\[ B[x^s; \rho^s] \cong B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s]/(T^{p_j} - x^{sp_j})B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s]. \]

Since \( B[x^s; \rho^s] \) and \( B[x^{sp_j}; \rho^{sp_j}] \) are \( H \)-separable extension over \( B \), it follows from [9, Proposition 2.2] that \( T^{p_j} - x^{sp_j} \) is an \( H \)-separable polynomial in \( B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s] \). We shall show that the center of \( B[x^{sp_j}; \rho^{sp_j}] = Z^{sp_j} \). In fact, the center of \( B[x^{sp_j}; \rho^{sp_j}] \supseteq Z^{sp_j} \) is clear and for any \( y = \sum_{\nu=0}^{q-1}(x^{sp_j})^\nu d_\nu \) in the center of \( B[x^{sp_j}; \rho^{sp_j}] \), we have

\[ (\rho^{sp_j})^\nu(b) d_\nu = d_\nu b \quad (b \in B) \quad \text{and} \quad \rho^{sp_j}(d_\nu) = d_\nu \quad (0 \leq \nu \leq q-1). \]

Since \( Y^q - u \) is an \( H \)-separable polynomial in \( B[Y; \rho^t] = B[Y; \rho^{sp_j}] \), it follows from [3, Lemma 1(l)] that \( d_\nu = 0 \) \((1 \leq \nu \leq q-1)\). Hence \( y = d_0 \in Z^{sp_j} \). Since \( T^{p_j} - x^{sp_j} \) is \( H \)-separable in \( B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s] \) and \( p_j \) is a prime number, \( Z^{sp_j}/Z^s \) is a \((\rho^s)^{Z^{sp_j}}\)-Galois extension of order \( p_j \) by [4, Theorem 2]. Thus the assertion follows from Lemma 2.

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In the proof of [4, Theorem 4] we have proved the following: Let \( f = X^{p^e} - u \) be a separable polynomial in \( B[X; \rho] \). If \( p \) is a prime number, and \( p \) is contained in the Jacobson radical \( J(B) \) of \( B \), then \( Z/Z^p \) is a \((\rho|Z)\)-Galois extension, and the group \((\rho|Z)\) is of order \( p \). We shall generalize this result as follows:

**Theorem 3.** Let \( f = X^m - u \) be in \( B[X; \rho] \) with \( fB[X; \rho] = B[X; \rho]_f \), and \( m = \ell p^e \), \( (\ell, p) = 1 \). Assume that \( p \) is a prime number, and \( p \) is contained in the Jacobson radical \( J(B) \) of \( B \).

1. If \( f \) is a separable polynomial in \( B[X; \rho] \), then \( Z/Z^{p^e} \) is a \((\rho|Z)\)-Galois extension, and the group \((\rho|Z)\) is of order \( p^e \).
2. If \( f \) is an \( H \)-separable polynomial in \( B[X; \rho] \) and \( \ell \) is a prime number, then \( Z/Z^{p^e} \) is a \((\rho|Z)\)-Galois extension and the group \((\rho|Z)\) is of order \( m \).

**Proof.** (1) Since \( f \) is a separable polynomial in \( B[X; \rho] \), it follows from [6, Theorem 3.1] that there exists an element \( c \in Z \) such that

\[
c + \rho(c) + \rho^2(c) + \cdots + \rho^{m-1}(c) = 1.
\]

We put here

\[
d = c + \rho(c) + \cdots + \rho^{\ell-1}(c).
\]

Then we have

\[
d + \rho'(d) + (\rho')^2(d) + \cdots + (\rho')^{p^e-1}(d) = 1.
\]

We consider the polynomial \( g = Y^{p^e} - u \in B[Y; \rho'] \). Then \( g \) is a separable polynomial in \( B[Y; \rho'] \) by [6, Theorem 3.1] again. Since \( p \in J(B) \), it follows from the proof of [4, Theorem 4] that \( Z/Z^{p^e} \) is a \((\rho'|Z)\)-Galois extension and the group \((\rho'|Z)\) is of order \( p^e \).

(2) We have

\[
B[x; \rho] \supset B[x'; \rho'] \cong B[Y; \rho']/(Y^{p^e} - u)B[Y; \rho'] \supset B.
\]

As was shown in (1), \( Y^{p^e} - u \) is an \( H \)-separable polynomial in \( B[Y; \rho'] \). Since \( B[x; \rho] \) is \( H \)-separable over \( B \), it follows from [9, Proposition 2.2] that \( B[x; \rho] \) is \( H \)-separable over \( B[x'; \rho'] \). Naturally, we can extend \( \rho \) to the automorphism \( \tilde{\rho} \) of \( B[x'; \rho'] \). Consider the skew polynomial ring \( B[x'; \rho'][T; \tilde{\rho}] \), where \( aT = T\tilde{\rho}(a) \). If \( B[x'; \rho'] \) is a \((\rho'|Z)\)-Galois extension.

\[
B[x; \rho] \cong B[x'; \rho'][T; \tilde{\rho}]/(T^{p^e} - x')B[x'; \rho'][T; \tilde{\rho}].
\]
$T' - x'$ is an $H$-separable polynomial in $B[x'; \rho'][T'; \tilde{\rho}]$. We shall show that $V_{B[x'; \rho']}(B[x'; \rho']) \cong Z^{*'}$. $V_{B[x'; \rho']}(B[x'; \rho']) \cong Z^{*'}$ is clear. On the other hand, for any $y = \sum_{\nu=0}^{\nu} (x')^\nu a_\nu \in V_{B[x'; \rho']}(B[x'; \rho'])$, we obtain

$$(\rho')^\nu(b)a_\nu = a_\nu b \quad (b \in B) \quad \text{and} \quad \rho'(a_\nu) = a_\nu \quad (0 \leq \nu \leq \rho^*-1).$$

Since $Y^{p^\nu} - u$ is an $H$-separable polynomial in $B[Y; \rho']$, it follows from [3, Lemma 1(1)] that $a_0 = 0 \quad (1 \leq \nu \leq \rho^*-1)$. Hence $y = a_0 \in Z^{*'}$, and so $V_{B[x'; \rho']}(B[x'; \rho']) = Z^{*'}$. Since $\ell$ is a prime number, it follows from [4, Theorem 2] that $Z^{*'}[Z^{*}]$ is a $(\rho[Z^{*}])$-Galois extension, and the group $(\rho[Z^{*}])$ is of order $\ell$. By (1), $Z[Z^{*}]$ is a $(\rho[Z])$-Galois extension, and the group $(\rho[Z])$ is of order $\rho^*$. Then the assertion follows from Lemma 2.

Combining Theorem 3 and [2, Proposition 1.4] we have the following which is a generalization of [4, Theorem 4].

**Corollary 4.** Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$, and $m = \ell p^s$, $(\ell, p) = 1$. Assume that $p$ is a prime number, and $p$ is contained in the Jacobson radical $J(B)$ of $B$. If $f$ is a separable polynomial in $B[X; \rho]$, then $g = Y^{p^\nu} - u$ is an $H$-separable polynomial in $B[Y; \rho']$.

The following is a sharpening of [4, Theorem 4], which corresponds to the results of Nagahara [7, Theorems 1 and 2].

**Corollary 5.** Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$, and $m = \ell p^s$, $(\ell, p) = 1$. Assume that $p$ is a prime number, and $p$ is contained in the Jacobson radical $J(B)$ of $B$. Then the following are equivalent:

(a) $u$ is invertible in $B^s$, and $Z[Z^s]$ is a $G$-Galois extension, where $G$ is the group generated by $\rho[Z]$ of order $m$.

(b) $B[x'; \rho']$ is an $H$-separable extension over $B$ for every divisor $n$ of $m$.

(c) $B[x; \rho]$ is a separable extension over $B$. $B[x; \rho]$ is an $H$-separable extension over $B[x'; \rho']$ and $B[x'; \rho']$ is an $H$-separable extension over $B$ for every divisor $r \ (1 < r < \ell)$ of $\ell$.

(d) $B[x; \rho]$ is a separable extension over $B$ and $B[x'; \rho']$ is an $H$-separable extension over $B[x'; \rho']$ for every divisor $r \ (1 \leq r \leq \ell)$ of $\ell$.

**Proof.** (a) $\iff$ (b) was shown in Theorem 1.

(b) $\implies$ (c). Since both $B[x; \rho]$ and $B[x'; \rho']$ are $H$-separable extension over $B$, it follows from [9, Proposition 2.2] that $B[x; \rho]$ is an $H$-separable extension over $B[x'; \rho']$. 

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(c) $\implies$ (d). Since $f$ is a separable polynomial in $B[X; \rho]$, as was shown in the proof of Theorem 3(1), $B[x^r; \rho^r]$ is an $H$-separable extension over $B$. Hence by [9, Proposition 2.2], $B[x^r; \rho^r]$ is an $H$-separable extension over $B[x^r; \rho^r]$ ($1 \leq r \leq \ell$).

(d) $\implies$ (a). It follows from Theorem 1, Theorem 3(1) and careful reading of the proof of Theorem 3(2).

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