Strong Convergence Theorems for Nonexpansive Mappings by Viscosity Approximation Methods in Banach Spaces

Xiaolong Qin∗  Yongfu Su†

Changqun Wu‡

∗Tianjin Polytechnic University
†Tianjin Polytechnic University
‡Henan University

Copyright ©2008 by the authors. Mathematical Journal of Okayama University is produced by The Berkeley Electronic Press (bepress). http://escholarship.lib.okayama-u.ac.jp/mjou
Strong Convergence Theorems for Nonexpansive Mappings by Viscosity Approximation Methods in Banach Spaces

Xiaolong Qin, Yongfu Su, and Changqun Wu

Abstract


KEYWORDS: Nonexpansive map; Iteration scheme; Sunny and nonexpansive retraction; viscosity method
STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS BY VISCOSITY APPROXIMATION METHODS IN BANACH SPACES

XIAOLONG QIN, YONGFU SU AND CHANGQUN WU


1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $J$ denotes the normalized duality mapping from $E$ into $2^E^*$ given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that a self mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in C.$$

We use $\Pi_C$ to denote the collection of all contractions on $C$. That is, $\Pi_C = \{f | f : C \to C \text{ a contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in $C$. Also, recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. Given a real number $t \in (0, 1)$ and a contraction $f \in \Pi_C$. We define a mapping $T_t x = tf(x) + (1 - t)Tx, \quad x \in C$. It is obviously that $T_t$ is a contraction on
C. In fact, for \( x, y \in C \), we obtain
\[
\|T_t x - T_t y\| \leq \|t(f(x) - f(y)) + (1 - t)(Tx - Ty)\|
\leq \alpha t\|x - y\| + (1 - t)\|Tx - Ty\|
\leq \alpha t\|x - y\| + (1 - t)\|x - y\|
= (1 - t(1 - \alpha))\|x - y\|.
\]

Let \( x_t \) be the unique fixed point of \( T_t \). That is, \( x_t \) is the unique solution of the fixed point equation
\[
(1.1) \quad x_t = tf(x_t) + (1 - t)Tx_t.
\]

A special case has been considered by Browder [1] in a Hilbert space as follows. Fix \( u \in C \) and define a contraction \( S_t \) on \( C \) by
\[
S_t x = tu + (1 - t)Tx, \quad x \in C.
\]

If we use \( z_t \) to denote the unique fixed point of \( S_t \), which yields that \( z_t = tu + (1 - t)Tz_t \).

In 1967, Browder [1] proved the following theorem.

**Theorem 1.1** In a Hilbert space, as \( t \to 0 \), \( z_t \) converges strongly to a fixed point of \( T \) that is closest to \( u \), that is, the nearest point projection of \( u \) onto \( F(T) \).

Also, In 1967, Halpern [5] firstly introduced this iteration scheme
\[
(1.2) \quad \begin{cases} 
  x_0 = x \in C \text{ chosen arbitrarily,} \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,
\end{cases}
\]
which is the special cases of
\[
(1.3) \quad \begin{cases} 
  x_0 = x \in C \text{ chosen arbitrarily,} \\
  x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n.
\end{cases}
\]

In [9], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If \( H \) is a Hilbert space, \( T : C \to C \) is a nonexpansive self-mapping on a nonempty closed convex \( C \) of \( H \) and \( f : C \to C \) is a contraction, he proved the following theorems.

**Theorem 1.2** (Moudafi [9]). The sequence \( \{x_n\} \) generated by the scheme
\[
x_n = \frac{1}{1 + \epsilon_n}Tx_n + \frac{\epsilon_n}{1 + \epsilon_n}f(x_n)
\]
converges strongly to the unique solution of the variational inequality:
\[
\bar{x} \in F(T), \quad \text{such that } \langle (I - f)\bar{x}, \bar{x} - x \rangle \leq 0, \quad \forall x \in F(T),
\]
where \( \{\epsilon_n\} \) is a sequence of positive numbers tending to zero.
**Theorem 1.3** (Moudafi [9]). With an initial $z_0 \in C$ defined the sequence \{z_n\} by

$$z_{n+1} = \frac{1}{1 + \epsilon_n} Tz_n + \frac{\epsilon_n}{1 + \epsilon_n} f(z_n).$$

Supposed that $\lim_{n \to \infty} \epsilon_n = 0$, and $\sum_{n=1}^{\infty} \epsilon = \infty$ and $\lim_{n \to \infty} \left| \frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n} \right| = 0$. Then \{z_n\} converges strongly to the unique solution of the variational inequality:

$$\bar{x} \in F(T) \text{ such that } \langle (I - f)\bar{x}, \bar{x} - x \rangle \leq 0, \forall x \in F(T).$$


**Theorem 1.4** (Xu [14]). Let $E$ be a uniformly smooth Banach space, $C$ a closed convex subset of $E$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$. Then the path \{x_t\} defined by $x_t = tf(x_t) + (1 - t)Tx_t$, $t \in (0, 1)$, converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by $Q(f) = \lim_{t \to 0} x_t$, the $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), j(Q(f) - x) \rangle, f \in \Pi_C, x \in F(T).$$

**Theorem 1.5** (Xu [14]). Let $E$ be a uniformly smooth Banach space, $C$ a closed convex subset of $E$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Assume that $\alpha_n \in (0, 1)$ satisfies the following conditions

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ or $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$. Then the sequence \{x_n\} generated by

$$x_0 \in C, \quad x_{n+1} = \alpha_nf(x_n) + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots$$

converges strongly to a fixed point of $T$.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [8] and is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where the initial guess $x_0$ is taken in $C$ arbitrarily and the sequence \{\alpha_n\}_{n=0}^{\infty} is in the interval $[0, 1]$. 

Qin et al.: Strong Convergence Theorems for Nonexpansive Mappings by
The second iteration process is referred to as Ishikawa’s iteration process [6] which is defined recursively by

\[
\begin{align*}
    y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\
    x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)Ty_n,
\end{align*}
\]

where the initial guess \(x_0\) is taken in \(C\) arbitrarily, \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in the interval \([0, 1]\). But both (1.4) and (1.5) have only weak convergence, in general (see [4] for an example). For example, Reich [11], shows that if \(E\) is a uniformly convex and has a Fréchet differentiable norm and if the sequence \(\{\alpha_n\}\) is such that \(\alpha_n(1 - \alpha_n) = \infty\), then the sequence \(\{x_n\}\) generated by processes (1.4) converges weakly to a point in \(F(T)\). (An extension of this result to processes (1.5) can be found in [13].) Therefore, many authors attempt to modify (1.4) and (1.5) to have strong convergence. Recently, Kim and Xu [7] introduced the following iteration process in the framework of Banach spaces.

\[
\begin{align*}
    z_n &= \gamma_n x_n + (1 - \gamma_n)T_2x_n, \\
    y_n &= \beta_n x_n + (1 - \beta_n)T_1z_n, \\
    x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n,
\end{align*}
\]

where the sequence \(\{\alpha_n\}\) in \((0, 1)\) and \(\{\beta_n\}\), \(\{\gamma_n\}\) are sequences in \([0, 1]\). We prove, under certain appropriate assumptions on the sequences \(\{\alpha_n\}\), \(\{\beta_n\}\) and \(\{\gamma_n\}\), that \(\{x_n\}\) defined by (1.7) converges to a common fixed point of \(T_1\) and \(T_2\), which solves some variational inequality.
If \( \{\gamma_n\} = 1 \) in (1.7) this can be viewed as a modified Mann iteration process

\[
\begin{align*}
  y_n &= \beta_n x_n + (1 - \beta_n) T_1 x_n, \\
  x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n.
\end{align*}
\]

If \( \{\gamma_n\} = 1 \) and \( \{\beta_n\} = 0 \) in (1.7), then (1.7) reduces to (1.3) which considered by Xu [14].

It is our purpose in this paper is to introduce this composite iteration scheme for approximating a common fixed point of two nonexpansive mappings by using viscosity methods in the framework of uniformly smooth Banach spaces. we establish the strong convergence of the sequence \( \{x_n\} \) defined by (1.7). Our results improve and extend the ones announced by Kim and Xu [7], Xu [14] and some others.

We need the following definitions and lemmas for the proof of our main results.

The norm of \( E \) is said to be Gâteaux differentiable (and \( E \) is said to be smooth) if

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for each \( x, y \) in its unit sphere \( U = \{x \in E : \|x\| = 1\} \). It is said to be uniformly Fréchet differentiable (and \( E \) is said to be uniformly smooth) if the limit in (1.9) is attained uniformly for \( (x, y) \in U \times U \).

**Lemma 1.1** A Banach space \( E \) is uniformly smooth if and only if the duality map \( J \) is single-valued and norm-to-norm uniformly continuous on bounded sets of \( E \).

**Lemma 1.2** In a Banach space \( E \), there holds the inequality

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad x, y \in E
\]

where \( j(x + y) \in J(x + y) \).

**Lemma 1.3** (Xu [15], [16]). Let \( \{\alpha_n\} \) be a sequence of nonnegative real numbers satisfying the property

\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n \geq 0,
\]

where \( \{\gamma_n\}_{n=0}^\infty \subset (0, 1) \) and \( \{\sigma_n\}_{n=0}^\infty \) such that

(i) \( \lim_{n \to \infty} \gamma_n = 0 \) and \( \sum_{n=0}^\infty \gamma_n = \infty \),

(ii) either \( \lim \sup_{n \to \infty} \sigma_n \leq 0 \) or \( \sum_{n=0}^\infty |\gamma_n \sigma_n| < \infty \).

Then \( \{\alpha_n\}_{n=0}^\infty \) converges to zero.
Recall that if $C$ and $D$ are nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny ([2], [12]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [2, 3, 12]: if $E$ is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx) \rangle \leq 0 \text{ for all } x \in C \text{ and } y \in D.$$  

Reich [10] showed that if $E$ is uniformly smooth and if $D$ is the fixed point set of a nonexpansive mapping from $C$ into itself, then there is a sunny nonexpansive retraction from $C$ onto $D$ and it can be constructed as follows.

**Lemma 1.4** (Reich [10]). Let $E$ be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converging strongly as $t \to 0$ to a fixed point of $T$. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t \to 0} x_t$. Then $Q$ is the unique sunny nonexpansive retract from $C$ onto $F(T)$; that is, $Q$ satisfies the property

$$\langle u - Qu, J(z - Qu) \rangle \leq 0, \quad u \in C, \quad z \in F(T).$$

**Lemma 1.5** (Xu [14]). Let $E$ be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converging strongly as $t \to 0$ to a fixed point of $T$. Define $Q : \Pi_C \to F(T)$ by

$$(1.10) \quad Qf = s - \lim_{t \to 0} x_t, \quad f \in \Pi_C.$$ 

Then $Q(f)$ solves the variational inequality

$$(1.11) \quad \langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

In particular, if $f = u$ is a constant, then (1.10) is reduced to the sunny nonexpansive retract from $C$ onto $F(T)$:

$$(1.12) \quad \langle u - Qu, J(p - Qu) \rangle \leq 0, \quad u \in C, \quad p \in F(T).$$

2. Main Results

**Theorem 2.1** Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$ and let $T_1, T_2 : C \to C$ be a pair of nonexpansive mappings such that $F(T_1T_2) = F(T_1) \cap F(T_2) \neq \emptyset$. The initial guess $x_0 \in C$ is chosen
arbitrarily and given sequences \( \{\alpha_n\}_{n=0}^\infty \) in \((0,1)\) and \( \{\beta_n\}_{n=0}^\infty \) and \( \{\gamma_n\}_{n=0}^\infty \) in \([0,1]\), the following conditions are satisfied

(i) \( \sum_{n=0}^\infty \alpha_n = \infty \), \( \alpha_n \to 0 \);
(ii) \( \beta_n \to 0 \), \( \gamma_n \to \beta \);
(iii) \( \sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty \), \( \sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty \) and \( \sum_{n=0}^\infty |\gamma_{n+1} - \gamma_n| \leq \infty \).

Let \( \{x_n\}_{n=1}^\infty \) be the composite process defined by

\[
\begin{align*}
    z_n &= \gamma_n x_n + (1 - \gamma_n) T_2 x_n, \\
    y_n &= \beta_n x_n + (1 - \beta_n) T_1 z_n, \\
    x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n.
\end{align*}
\]

Then \( \{x_n\}_{n=1}^\infty \) converges strongly to some common fixed point \( p \in F(T_1) \cap F(T_2) \) which solves the variational inequality

\[
(2.1) \quad \langle (I - f) Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T_1) \cap F(T_2).
\]

**Proof.** First we observe that \( \{x_n\}_{n=0}^\infty \) is bounded. Indeed, taking a fixed point \( p \) of \( F(T_1) \cap F(T_2) \), we note that

\[
(2.2) \quad \|z_n - p\| \leq \gamma_n \|x_n - p\| + (1 - \gamma_n) \|T_2 x_n - p\| \leq \|x_n - p\|.
\]

It follows that

\[
(2.3) \quad \|y_n - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T_1 z_n - p\| \\
         \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|z_n - p\| \\
         \leq \|x_n - p\|.
\]

It follows from (2.3) that

\[
\|x_{n+1} - p\| \leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n) \|y_n - p\| \\
      \leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\
      \leq \max\{\frac{1}{1 - \alpha} \|f(p) - p\|, \|x_n - p\|\}.
\]

Now, an induction yields

\[
(2.4) \quad \|x_n - p\| \leq \max\{\frac{1}{1 - \alpha} \|f(p) - p\|, \|x_0 - p\|\}. \quad n \geq 0,
\]

which implies that \( \{x_n\} \) is bounded, so are \( \{T_2 x_n\} \), \( \{f(x_n)\} \), \( \{y_n\} \) and \( \{T_1 z_n\} \).

Since condition (i), we obtain

\[
(2.6) \quad \|x_{n+1} - y_n\| = \alpha_n \|f(x_n) - y_n\| \to 0, \quad as \ n \to \infty.
\]

Next, we claim that

\[
(2.6) \quad \|x_{n+1} - x_n\| \to 0.
\]
In order to prove (2.6) from

\[
\begin{aligned}
 x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \\
x_n &= \alpha_{n-1} f(x_n) + (1 - \alpha_{n-1})y_n.
\end{aligned}
\]

We have

\[
x_{n+1} - x_n = (1 - \alpha_n)(y_n - y_{n-1}) + (\alpha_n - \alpha_{n-1})(y_{n-1} - f(x_{n-1})) + \alpha_n(f(x_n) - f(x_{n-1})).
\]

It follows that

(2.7) \[|x_{n+1} - x_n| \leq (1 - \alpha_n)|y_n - y_{n-1}| + |\alpha_n - \alpha_{n-1}||y_{n-1} - f(x_{n-1})| + \alpha_n|x_n - x_{n-1}|.\]

Similarly, Since

\[
\begin{aligned}
y_n &= \beta_n x_n + (1 - \beta_n)T_1 z_n, \\
y_{n-1} &= \beta_{n-1} x_{n-1} + (1 - \beta_{n-1})T_1 z_{n-1}.
\end{aligned}
\]

We obtain

\[
y_n - y_{n-1} = (1 - \beta_n)(T_1 z_n - T_1 z_{n-1}) + \beta_n(x_n - x_{n-1}) + (T_1 z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n).
\]

It follow that

(2.8) \[|y_n - y_{n-1}| \leq (1 - \beta_n)|T_1 z_n - T_1 z_{n-1}| + \beta_n|x_n - x_{n-1}| + \|T_1 z_{n-1} - x_{n-1}||\beta_{n-1} - \beta_n|.
\]

On the other hand, from

\[
\begin{aligned}
z_n &= \gamma_n x_n + (1 - \gamma_n)T_2 x_n, \\
z_{n-1} &= \gamma_{n-1} x_{n-1} + (1 - \gamma_{n-1})T_2 z_{n-1},
\end{aligned}
\]

we also can obtain

\[
z_n - z_{n-1} = (1 - \gamma_n)(T_2 x_n - T_2 x_{n-1}) + \gamma_n(x_n - x_{n-1}) + (\gamma_{n-1} - \gamma_n)(T_2 x_{n-1} - x_{n-1}),
\]

which yields that

(2.9) \[|z_n - z_{n-1}| \leq |x_n - x_{n-1}| + |\gamma_{n-1} - \gamma_n||T_2 x_{n-1} - x_{n-1}|.
\]

Substituting (2.9) into (2.8), we get

(2.10) \[|y_n - y_{n-1}| \leq (1 - \beta_n)(|x_n - x_{n-1}| + |\gamma_{n-1} - \gamma_n||T_2 x_{n-1} - x_{n-1}|) + \beta_n|x_n - x_{n-1}| + \|T_1 z_{n-1} - x_{n-1}||\beta_{n-1} - \beta_n|.
\]
Similarly, substitute (2.11) into (2.7) yields that
\[
\|x_{n+1} - x_n\| \leq (1 - \alpha_n)(\|x_n - x_{n-1}\| + |\gamma_n - \gamma_n|\|T_2x_{n-1} - x_{n-1}\|
\]
\[
+ \|T_1z_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n| + |\alpha_{n-1} - \alpha_n||y_n - f(x_{n-1})| + \alpha_{n-1}\|x_n - x_{n-1}\|
\]
\[
\leq (1 - (1 - \alpha)\alpha_n)\|x_n - x_{n-1}\|
\]
\[
+ M_1(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n| + |\gamma_{n-1} - \gamma_n|),
\]
where \(M_1\) is a constant such that
\[
M_1 \geq \max\{\|y_n - f(x_{n-1})\|, \|x_{n-1} - T_2x_{n-1}\|, \|x_{n-1} - T_1z_{n-1}\|\}
\]
for all \(n\). By assumptions (i)-(iii), we have that
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} (1 - \alpha)\alpha_n = \infty,
\]
and
\[
\sum_{n=1}^{\infty} (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) < \infty.
\]
Hence, Lemma 1.3 is applicable to (2.12) and we obtain (2.6) holds. Observe that
\[
\|T_1 T_2 x_n - x_n\|
\]
\[
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_1 z_n\| + \|T_1 z_n - T_1 T_2 x_n\|
\]
\[
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - T_1 z_n\| + \|z_n - T_2 x_n\|
\]
\[
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - T_1 z_n\| + \gamma_n \|x_n - T_2 x_n\|.
\]
Since assumption \(\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = 0\), (2.5) and (2.6), we know
\[
\|T_1 T_2 x_n - x_n\| \to 0.
\]
Put \(T = T_1 T_2\). Since \(T_1\) and \(T_2\) are nonexpansive, we have \(T\) is also nonexpansive. Next, we claim that
\[
\limsup_{n \to \infty} (f(q) - q, J(x_n - q)) \leq 0,
\]
where \(q = Qf = \lim_{t \to 0} x_t\) with \(x_t\) being the fixed point of the contraction \(x \mapsto tf(x) + (1 - t)Tx\), where \(T = T_1 T_2\). From \(x_t\) solves the fixed point
equation
\[ x_t = tf(x_t) + (1-t)Tx_t. \]
Thus we have
\[ \|x_t - x_n\| = \|(1-t)(Tx_t - x_n) + t(f(x_t) - x_n)\|. \]
It follows from Lemma 1.2 that
\[ \|x_t - x_n\|^2 \leq (1-t)^2\|Tx_t - x_n\|^2 + 2t(f(x_t) - x_n, J(x_t - x_n)) \]
\[ + 2t^2\|f(x_t) - x_t, J(x_t - x_n)\| + 2t\|x_t - x_n\|^2, \]
where
\[ f_n(t) = (2\|x_t - x_n\| + \|x_n - Tx_n\|)|x_n - Tx_n| \to 0, \text{ as } n \to 0. \]
It follows that
\[ \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \]
Let \( n \to \infty \) in (2.18) and note (2.17) yields
\[ \limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{t}{2}M_2, \]
where \( M_2 > 0 \) is a constant such that \( M_2 \geq \|x_t - x_n\|^2 \) for all \( t \in (0, 1) \) and \( n \geq 1 \). Taking \( t \to 0 \) from (2.19), we have
\[ \limsup_{n \to \infty} \limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq 0. \]
So, for any \( \epsilon > 0 \), there exists a positive number \( \delta_1 \) such that, for \( t \in (0, \delta_1) \), we get
\[ \limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{\epsilon}{2}. \]
On the other hand, since \( x_t \to q \) as \( t \to 0 \), from Lemma 1.1, there exists \( \delta_2 > 0 \) such that, for \( t \in (0, \delta_2) \) we have
\[ \langle f(q) - q, J(x_n - q) - J(x_n - x_t) \rangle - \langle x_t - f(x_t), J(x_t - x_n) \rangle \]
\[ \leq \langle f(q) - q, J(x_n - q) - J(x_n - x_t) \rangle - \langle f(q) - q, J(x_n - x_t) \rangle \]
\[ + \langle f(q) - q, J(x_n - x_t) \rangle - \langle x_t - f(x_t), J(x_t - x_n) \rangle \]
\[ \leq \|f(q) - q\| \|J(x_n - q) - J(x_n - x_t)\| \]
\[ + \|f(q) - f(x_t) - q + x_t, J(x_n - q)\| \]
\[ \leq \|f(q) - q\| \|J(x_n - q) - J(x_n - x_t)\| \]
\[ + \|f(q) - f(x_t) - q + x_t\| \|x_n - q\| \]
\[ < \frac{\epsilon}{2}. \]
Picking \( \delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta) \), we have
\[ \langle f(q) - q, J(x_n - q) \rangle \leq \langle x_t - f(x_t), J(x_t - x_n) \rangle + \frac{\epsilon}{2}. \]
STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS

That is,
\[
\limsup_{n \to \infty} \langle f(q) - q, J(x_n - q) \rangle \leq \limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle + \frac{\epsilon}{2}.
\]
It follows from (2.21) that
\[
\limsup_{n \to \infty} \langle f(q) - q, J(x_n - q) \rangle \leq \epsilon.
\]
Since \(\epsilon\) is chosen arbitrarily, we have
(2.21).
\[
\limsup_{n \to \infty} \langle f(q) - q, J(x_n - q) \rangle \leq 0
\]
Finally, we show that \(x_n \to q\) strongly and this concludes the proof. Indeed, using Lemma 1.2 again we obtain
\[
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(y_n - q) + \alpha_n(f(x_n) - q)\|^2
\]
\[
\leq (1 - \alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, J(x_{n+1} - q) \rangle
\]
\[
\leq (1 - \alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\|x_n - q\|\|x_{n+1} - q\|
\]
\[
+ 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle
\]
\[
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\|x_n - q\|^2 + \|x_{n+1} - q\|^2)
\]
\[
+ 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle.
\]
Therefore, we obtain
\[
\|x_{n+1} - q\|^2
\]
\[
\leq \frac{1 - 2(1 - \alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n}\|x_n - q\|^2 - \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(q) - q, J(x_{n+1} - q) \rangle
\]
\[
\leq \frac{1 - 2(1 - \alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n}\|x_n - q\|^2 - \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(q) - q, J(x_{n+1} - q) \rangle + M_2\alpha_n^2
\]
\[
= (1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n})\|x_n - q\|^2
\]
\[
+ \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}(\frac{M_2(1 - \alpha\alpha_n)\alpha_n}{2(1 - \alpha)} + \frac{1}{1 - \alpha}\langle f(q) - q, J(x_{n+1} - q) \rangle).
\]
Now we apply Lemma 1.3 and use (2.21) to see that \(\|x_n - q\| \to 0\). This completes the proof.

As corollaries of Theorem 2.1, we have the following.

**Corollary 2.2** Let \(C\) be a closed convex subset of a uniformly smooth Banach space \(E\) and let \(T_1 : C \to C\) be a nonexpansive mapping such that
\( F(T_1) \neq \emptyset \). The initial guess \( x_0 \in C \) is chosen arbitrarily and given sequences \( \{\alpha_n\}_{n=0}^{\infty} \) in \((0,1)\) and \( \{\beta_n\}_{n=0}^{\infty} \) in \([0,1]\), the following conditions are satisfied

(i) \( \sum_{n=0}^{\infty} \alpha_n = \infty, \ \alpha_n \to 0; \)

(ii) \( \beta_n < a, \) for some \( a \in [0,1); \)

(iii) \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty. \)

Let \( \{x_n\}_{n=1}^{\infty} \) be the composite process defined by (1.8), then \( \{x_n\}_{n=1}^{\infty} \) converges strongly to some fixed point \( p \in F(T_1) \) which \( Q(f) \) solves the variational inequality

\[
\langle (I-f)Q(f), J(Q(f)-p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T_1).
\]

**Proof.** By taking \( \{\gamma_n\} = 1 \), we can obtain the desired conclusion. This completes the proof.

**Corollary 2.3** (Xu [14]). Let \( E \) be a uniformly smooth Banach space, \( C \) a closed convex subset of \( E \) and \( T : C \to C \) a nonexpansive mapping with \( F(T) \neq \emptyset, \) and \( f \in \Pi_C. \) Assume that \( \alpha_n \in (0,1) \) satisfies the following conditions

(i) \( \lim_{n \to \infty} \alpha_n = 0; \)

(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty; \)

(iii) \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty. \) Then the sequence \( \{x_n\} \) generated by

\[
x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots
\]

converges strongly to \( Q(f) \), which solves the variational inequality

\[
\langle (I-f)Q(f), J(Q(f)-p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T).
\]

**Proof.** By taking \( \{\gamma_n\} = 1 \) and \( \{\beta_n\}=0 \), we can obtain the desired conclusion. This completes the proof.

**REFERENCES**


XIAOLONG QIN
DEPARTMENT OF MATHEMATICS, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300160,
PR CHINA
e-mail address: qxlxajh@163.com

YONGFU SU
DEPARTMENT OF MATHEMATICS, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300160,
PR CHINA
e-mail address: suyongfu@tjpu.edu.cn

CHANGQUN WU
SCHOOL OF BUSINESS AND ADMINISTRATION, HENAN UNIVERSITY, KAIFENG 475001,
PR CHINA
e-mail address: kyls2003@yahoo.com.cn

(Received December 7, 2006)