On $\Phi$-recurrent $N(k)$-contact Metric Manifolds

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Abstract

In this paper we prove that a $\Phi$-recurrent $N(k)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients. Next, we prove that a 3-dimensional $\Phi$-recurrent $N(k)$-contact metric manifold is of constant curvature. The existence of a $\Phi$-recurrent $N(k)$-contact metric manifold is also proved.

KEYWORDS: N(k)-contact metric manifolds, eta-Einstein manifold, Phi-recurrent N(k)-contact metric manifolds
ON $\Phi$ -RECURRENT $N(k)$-CONTACT METRIC MANIFOLDS

Dedicated to PROFESSOR DAVID E. BLAIR

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Abstract. In this paper we prove that a $\phi$-recurrent $N(k)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients. Next, we prove that a 3-dimensional $\phi$-recurrent $N(k)$-contact metric manifold is of constant curvature. The existence of a $\phi$-recurrent $N(k)$-contact metric manifold is also proved.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T.Takahashi [1] introduced the notion of local $\phi$-symmetry on a Sasakian manifold. Generalizing the notion of local $\phi$-symmetry, one of the authors, De, [2] introduced the notion of $\phi$-recurrent Sasakian manifold. In the context of contact geometry the notion of $\phi$-symmetry is introduced and studied by Boeckx, Bueken and Vanhecke [3] with several examples.

In the present paper we study $\phi$-recurrent $N(k)$-contact metric manifold which generalizes the result of De, Shaikh and Biswas [2]. The paper is organized as follows:

Section 2 contains necessary details about contact metric manifolds, some preliminaries and a brief account of $(k,\mu)$ manifolds and the basic results. In Section 3, it is proved that a $\phi$-recurrent $N(k)$-contact metric manifold is a special type of $\eta$-Einstein manifold. Also it is shown that the characteristic vector field of the $N(k)$-contact metric manifold and the vector field associated to the 1-form of recurrence are co-directional. In Section 4, it is also proved that a 3-dimensional $\phi$-recurrent $N(k)$-contact metric manifold is of constant curvature. The last section provides the existence of the $\phi$-recurrent $N(k)$-contact metric manifold by an example which is neither symmetric nor locally $\phi$-symmetric.

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Key words and phrases. $N(k)$-contact metric manifolds, $\eta$-Einstein manifold, $\phi$-recurrent $N(k)$-contact metric manifolds.

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2. Contact Metric Manifolds

A \((2n+1)\)-dimensional manifold \(M^{2n+1}\) is said to admit an almost contact structure if it admits a tensor field \(\phi\) of type \((1,1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying
\[
(a) \quad \phi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \phi \xi = 0, \quad (d) \quad \eta \circ \phi = 0.
\]

An almost contact metric structure is said to be normal if the induced almost complex structure \(J\) on the product manifold \(M^{2n+1} \times \mathbb{R}\) defined by
\[
J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt})
\]
is integrable, where \(X\) is tangent to \(M\), \(t\) is the coordinate of \(\mathbb{R}\) and \(f\) is a smooth function on \(M \times \mathbb{R}\). Let \(g\) be a compatible Riemannian metric with almost contact structure \((\phi, \xi, \eta)\), that is,
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]
Then \(M\) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\phi, \xi, \eta, g)\). From (2.1) it can be easily seen that
\[
(a) g(X, \phi Y) = -g(\phi X, Y), \quad (b) g(X, \xi) = \eta(X),
\]
for all vector fields \(X, Y\). An almost contact metric structure becomes a contact metric structure if
\[
g(X, \phi Y) = d\eta(X, Y),
\]
for all vector fields \(X, Y\). The 1-form \(\eta\) is then a contact form and \(\xi\) is its characteristic vector field. We define a \((1,1)\) tensor field \(h\) by
\[
h = \frac{1}{2} \mathcal{L}_\xi \phi,
\]
where \(\mathcal{L}\) denotes the Lie-differentiation. Then \(h\) is symmetric and satisfies \(h\phi = -\phi h\). We have \(Tr:h = Tr:\phi = 0\) and \(h\xi = 0\). Also,
\[
\nabla_X \xi = -\phi X - \phi h X,
\]
holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if
\[
(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,
\]
where \(\nabla\) is the Levi-Civita connection of the Riemannian metric \(g\). A contact metric manifold \(M^{2n+1}(\phi, \xi, \eta, g)\) for which \(\xi\) is a Killing vector is said to be a \(K\)-contact manifold. A Sasakian manifold is \(K\)-contact but not conversely. However a 3-dimensional \(K\)-contact manifold is Sasakian [4]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a
contact metric structure satisfying $R(X,Y)\xi = 0$ ([5]). On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$  

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [6] considered the $(k, \mu)$-nullity condition on a contact metric manifold and gave several reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ ([6], [7]) of a contact metric manifold $M$ is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{W \in T_pM : R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\},$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold $M^{2n+1}$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-manifold. In particular on a $(k, \mu)$-manifold, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$

On a $(k, \mu)$-manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, the $(k, \mu)$-nullity condition determines the curvature of $M^{2n+1}$ completely [6]. In fact, for a $(k, \mu)$-manifold, the condition of being a Sasakian manifold, a $K$-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a $(k, \mu)$-manifold the following relations hold ([6], [8]):

$$h^2 = (k - 1)\phi^2, \quad k \leq 1,$$

$$\nabla_X \phi(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX],$$

$$S(X, \xi) = 2nk\eta(X),$$

$$S(X, Y) = [2(n - 1) - n\mu]g(X,Y) + [2(n - 1) + \mu]g(hX,Y) + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1,$$

$$r = 2n(2n - 2 + k - n\mu),$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX,Y),$$

where $S$ is the Ricci tensor of type $(0, 2)$, $Q$ is the Ricci-operator, that is, $g(QX,Y) = S(X,Y)$ and $r$ is the scalar curvature of the manifold. From (2.5), it follows that

$$\nabla_X \eta)(Y) = g(X + hX, \phi Y).$$
Also in a \((k, \mu)\)-manifold

\begin{equation}
\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\
\quad + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)]
\end{equation}

holds.

The \(k\)-nullity distribution \(N(k)\) of a Riemannian manifold \(M\) \[9\] is defined by

\[ N(k) : p \mapsto N_p(k) = \{ Z \in T_pM : R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \}, \]

\(k\) being a constant. If the characteristic vector field \(\xi \in N(k)\), then we call a contact metric manifold an \(N(k)\)-contact metric manifold \[10\]. If \(k = 1\), then \(N(k)\)-contact metric manifold is Sasakian and if \(k = 0\), then \(N(k)\)-contact metric manifold is locally isometric to the product \(E^{n+1} \times S^n(4)\) for \(n > 1\) and flat for \(n = 1\). If \(k < 1\), the scalar curvature is \(r = 2n(2n - 2 + k)\).

In \[11\], \(N(k)\)-contact metric manifold were studied in some detail. For more details we refer to \[12\] \[13\].

In \(N(k)\)-contact metric manifold the following relations hold:

\begin{align}
(2.18) & \quad h^2 = (k - 1)\phi^2, \quad k \leq 1, \\
(2.19) & \quad (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \\
(2.20) & \quad R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \\
(2.21) & \quad S(X, \xi) = 2nk\eta(X), \\
(2.22) & \quad S(X, Y) = 2(n - 1)g(X,Y) + 2(n - 1)g(hX,Y) \\
& \quad + [2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1, \\
(2.23) & \quad r = 2n(2n - 2 + k), \\
(2.24) & \quad S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX,Y), \\
(2.25) & \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y), \\
(2.26) & \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \\
(2.27) & \quad \eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\end{align}
3. $\phi$-recurrent $N(k)$-contact metric manifolds

**Definition 1.** ([1]) A Sasakian manifold is said to be locally $\phi$-symmetric if the relation

$$\phi^2((\nabla_W R)(X,Y)Z) = 0$$

holds for all vector fields $X$, $Y$, $Z$, $W$ orthogonal to $\xi$.

**Definition 2.** ([2]) A $N(k)$-contact metric manifold is said to be $\phi$-recurrent if and only if there exists a non-zero 1-form $A$ such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z,$$

for all vector fields $X$, $Y$, $Z$, $W$. Here $X$, $Y$, $Z$, $W$ are arbitrary vector fields which are not necessarily orthogonal to $\xi$.

If the 1-form $A$ vanishes identically, then the manifold is said to be a locally $\phi$-symmetric manifold.

**Definition 3.** ([6]) A contact manifold is said to be $\eta$-Einstein if the Ricci tensor $S$ of type $(0,2)$ satisfies the condition

$$S(X,Y) = a g(X,Y) + b \eta(X)\eta(Y),$$

where $a$ and $b$ are smooth functions on $M^{2n+1}$.

Now we prove the main theorem of the paper.

**Theorem 3.1.** A $\phi$-recurrent $N(k)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients.

**Proof.** By virtue of (2.1)(a) and (3.1) we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z,$$

from which it follows that

$$-(\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U).$$

Let $\{e_i\}$, $i = 1, 2, 3, \ldots, 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = \{e_i\}$ in (3.4) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z).$$

The second term of (3.5) by putting $Z = \xi$ takes the form

$$g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi),$$

which is denoted by $E$. In this case $E$ vanishes.
We have
\[ g(\nabla_W R(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \]
\[ - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \]
for \( p \in M \). Using (2.3)(b) and (2.27) we obtain
\[ g(R(e_i, \nabla_W Y)\xi, \xi) = g(k[\eta(\nabla_W Y) e_i - \eta(e_i)\nabla_W Y], \xi) \]
\[ = k[\eta(\nabla_W Y) \eta(e_i) - \eta(e_i)\eta(\nabla_W Y)] = 0. \]

Thus we obtain
\[ g(\nabla_W R(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \]

In virtue of (3.7), (2.26) and (2.3)(a) we get
\[ g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \]
which implies
\[ g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) = 0. \]

Using (2.5) and applying skew-symmetry of \( R \) we get
\[ g((\nabla_W R)(e_i, Y)\xi, \xi) \]
\[ = g(R(e_i, Y)\xi, \phi W + \phi h W) + g(R(e_i, Y)(\phi W + \phi h W)\xi, \xi) \]
\[ = g(R(\phi W + \phi h W)\xi, \xi) + g(R(\phi W + \phi h W)Y, e_i). \]

Hence we obtain
\[ E = \sum_{i=1}^{2n+1} g(R(\phi W + \phi h W)\xi, \xi)g(\xi, e_i) \]
\[ + g(R(\xi, \phi W + \phi h W)Y, e_i)g(\xi, e_i) \]
\[ = g(R(\phi W + \phi h W)\xi, \xi) + g(R(\xi, \phi W + \phi h W)Y, \xi) = 0. \]

Replacing \( Z \) by \( \xi \) in (3.5) and using (2.21) we have
\[ -(\nabla_W S)(Y, \xi) = 2nkA(W)\eta(Y). \]

Now we have
\[ (\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \]

Using (2.21) and (2.5) in the above relation, it follows that
\[ (\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)(Y) + S(Y, \phi W + \phi h W). \]

In virtue of (3.7), (2.26) and (2.3)(a) we get
\[ (\nabla_W S)(Y, \xi) = -2nk(\phi W + \phi h W, Y) + S(Y, \phi W + \phi h W). \]
By (3.6) and (3.8) we have

\begin{equation}
2nkg(\phi W + \phi hW, Y) - S(Y, \phi W + \phi hW) = 2nkA(W)\eta(Y).
\end{equation}

Replacing \( Y \) by \( \phi Y \) in (3.9) and using (2.1)(d), (2.2), (2.25) we get

\[2nkg(\phi W + \phi hW, \phi Y) - S(\phi Y, \phi W + \phi hW) = 0\]

or,

\[2nkg(W + hW, Y) - \eta(W + hW)\eta(Y)] - S(Y, W + hW)
+ 2nk\eta(W + hW)\eta(Y) + 4(n - 1)g(hY, W + hW) = 0\]

or,

\[2nkg(Y, W) + 2nkg(Y, hW) - S(Y, W) - S(Y, hW)
+ 4(n - 1)g(Y, hW) + 4(n - 1)g(Y, hW^2) = 0\]

since, \( g(X, hY) = g(X, Y) \). Now by (2.23), (2.18) and (2.1)(a) this implies

\[S(Y, W) + S(Y, hW) = 2nkg(Y, W) + [2nk + 4(n - 1)]g(Y, hW)
+ 4(n - 1)(k - 1)g(Y, -W + \eta(W)\xi)\]

or,

\[S(Y, W) + 2(n - 1)g(Y, hW) - 2(n - 1)(k - 1)g(Y, W)
+ 2(n - 1)(k - 1)\eta(Y)\eta(W) = [2nk - 4(n - 1)(k - 1)]g(Y, W)
+ [2nk + 4(n - 1)]g(Y, hW) + 4(n - 1)(k - 1)\eta(Y)\eta(W),\]

which implies,

\begin{equation}
S(Y, W) = 2(n + k - 1)g(Y, W)
+ 2(nk + n - 1)g(Y, hW) + 2(n - 1)(k - 1)\eta(Y)\eta(W).
\end{equation}

Replacing \( W \) by \( hW \) and using (2.23), (2.18) and (2.1)(a) we get from (3.10)

\[-2kg(Y, hW) = -2nk(k - 1)g(Y, W) + 2nk(k - 1)\eta(Y)\eta(W)\]

Since we may assume that \( k \neq 0 \), this implies

\begin{equation}
g(Y, hW) = n(k - 1)g(Y, W) - n(k - 1)\eta(Y)\eta(W).
\end{equation}

From (3.10) and (3.11) we get

\[S(Y, W) = 2[(n + k - 1) + n(k - 1)(nk + n - 1)]g(Y, W)
+ 2[(n - 1)(k - 1) - n(k - 1)(nk + n - 1)]\eta(Y)\eta(W)\]

or,

\begin{equation}
S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W),
\end{equation}

where \( a = 2[(n + k - 1) + n(k - 1)(nk + n - 1)] \), \( b = 2[(n - 1)(k - 1) - n(k - 1)(nk + n - 1)] \) are constant. So, the manifold is an \( \eta \)-Einstein manifold with constant coefficients. Hence the theorem is proved. \( \Box \)
Now, from (3.3) we have
\[(3.13) \quad (\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z.\]
From (3.13) and the second Bianchi identity we get
\[(3.14) \quad A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0.\]
Using (2.28), we get from (3.14)
\[(3.15) \quad k[A(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + A(X)(g(W,Z)\eta(Y)
- g(Y,Z)\eta(W)) + A(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X))] = 0.\]
Putting \(Y = Z = \{e_i\}\) in (3.15) and taking summation over \(i, 1 \leq i \leq 2n+1\), we get
\[k(2n - 1)[A(W)\eta(X) - A(X)\eta(W)] = 0,\]
which implies that
\[(3.16) \quad A(W)\eta(X) = A(X)\eta(W).\]
Replacing \(X\) by \(\xi\) in (3.16), it follows that
\[(3.17) \quad A(W) = \eta(\rho)\eta(W),\]
for any vector field \(W\), where \(A(\xi) = g(\xi, \rho) = \eta(\rho)\), \(\rho\) being the vector field associated to the 1-form \(A\), that is, \(g(X, \rho) = A(X)\). Hence we can state the following theorem:

**Theorem 3.2.** In a \(\phi\)-recurrent \(N(k)\)-contact metric manifold \((M^{2n+1}, g)\), \(n > 1\), the characteristic vector field \(\xi\) and the vector field \(\rho\) associated to the 1-form \(A\) are co-directional and the 1-form \(A\) is given by (3.17).

4. 3-dimensional \(\phi\)-recurrent \(N(k)\)-contact metric manifolds

In a 3-dimensional Riemannian manifold we have
\[(4.1) \quad R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X
- S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X],\]
where \(Q\) is the Ricci-operator, that is, \(g(QX,Y) = S(X,Y)\) and \(r\) is the scalar curvature of the manifold. Now putting \(Z = \xi\) in (4.1) and using (2.3)(b) and (2.21), we get
\[(4.2) \quad R(X,Y)\xi = \eta(Y)QX - \eta(X)QY
+ 2k[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X].\]
Using (2.27) in (4.2), we have
\[(4.3) \quad (k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX.\]
Puting $Y = \xi$ in (4.3) and using (2.21), we get

\begin{equation}
QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi.
\end{equation}

Therefore, it follows from (4.4) that

\begin{equation}
S(X, Y) = (\frac{r}{2} - k)g(X, Y) + (3k - \frac{r}{2})\eta(X)\eta(Y).
\end{equation}

Thus from (4.1), (4.4) and (4.5), we get

\begin{equation}
R(X, Y)Z = (\frac{r}{2} - 2k)[g(Y, Z)X - g(X, Z)Y] + (3k - \frac{r}{2})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].
\end{equation}

Taking the covariant differentiation to the both sides of the equation (4.6), we get

\begin{equation}
(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\eta(X)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] + (3k - \frac{r}{2})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y](\nabla_W \xi) + (3k - \frac{r}{2})[\eta(Y)\eta(Z)X - \eta(X)\eta(Y)](\nabla_W \eta)(Z) + (3k - \frac{r}{2})[g(Y, Z)\xi - \eta(Z)Y](\nabla_W \eta)(X) - (3k - \frac{r}{2})[g(X, Z)\xi - \eta(Z)X](\nabla_W \eta)\eta(\xi).
\end{equation}

Noting that we may assume that all vector fields $X$, $Y$, $Z$, $W$ are orthogonal to $\xi$ and using (2.1)(b), we get

\begin{equation}
(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi] + (3k - \frac{r}{2})[g(Y, Z)\eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\eta(X)\xi] + (3k - \frac{r}{2})[\eta(Y)\eta(Z)X - \eta(X)\eta(Y)](\nabla_W \eta)(Z).
\end{equation}

Applying $\phi^2$ to the both sides of (4.8) and using (2.1)(a) and (2.1)(c), we get

\begin{equation}
\phi^2(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2}[g(X, Z)Y - g(Y, Z)X].
\end{equation}

By (3.1) the equation (4.9) reduces to

\begin{equation}
A(W)R(X, Y)Z = \frac{dr(W)}{2}[g(X, Z)Y - g(Y, Z)X].
\end{equation}
Putting \( W = \{ e_i \} \), where \( \{ e_i \} \), \( i = 1, 2, 3 \), is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \( i \), \( 1 \leq i \leq 3 \), we obtain

\[
R(X, Y)Z = \lambda [g(X, Z)Y - g(Y, Z)X],
\]

where \( \lambda = \frac{dr(e_i)}{2A(e_i)} \) is a scalar, since \( A \) is a non-zero 1-form. Then by Schur’s theorem \( \lambda \) will be a constant on the manifold. Therefore, \( M^3 \) is of constant curvature \( \lambda \). Thus we get the following theorem:

**Theorem 4.1.** A 3-dimensional \( \phi \)-recurrent \( N(k) \)-contact metric manifold is of constant curvature.

### 5. Existence of \( \phi \)-recurrent \( N(k) \)-contact metric manifolds

In this section we give an example of \( \phi \)-recurrent \( N(k) \)-contact metric manifold which is neither symmetric nor locally \( \phi \)-symmetric. We take the 3-dimensional manifold \( M = \{ (x, y, z) \in \mathbb{R}^3 : x \neq 0 \} \), where \( (x, y, z) \) are the standard coordinates in \( \mathbb{R}^3 \). Let \( \{ E_1, E_2, E_3 \} \) be linearly independent global frame on \( M \) given by

\[
E_1 = \frac{2}{x} \frac{\partial}{\partial y}, \quad E_2 = \frac{2}{x} \frac{\partial}{\partial x} - \frac{4z}{x} \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.
\]

Let \( g \) be the Riemannian metric defined by

\[
g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,
\]

\[
g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.
\]

Let \( \eta \) be the 1-form defined by \( \eta(U) = g(U, E_3) \) for any \( U \in \chi(M) \). Let \( \phi \) be the \( (1, 1) \) tensor field defined by \( \phi E_1 = E_2, \phi E_2 = -E_1, \phi E_3 = 0 \). Then using the linearity of \( \phi \) and \( g \) we have \( \eta(E_3) = 1, \phi^2 U = -U + \eta(U)E_3 \) and \( g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W) \) for any \( U, W \in \chi(M) \). Moreover \( hE_1 = -E_1, hE_2 = E_2 \) and \( hE_3 = 0 \). Thus for \( E_3 = \xi, (\phi, \xi, \eta, g) \) defines a contact metric structure on \( M \). Hence we have \( [E_1, E_2] = 2E_3 + \frac{2}{x}E_1, [E_1, E_3] = 0, [E_2, E_3] = 2E_1 \).

The Riemannian connection \( \nabla \) of the metric \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)
\]

\[
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Taking \( E_3 = \xi \) and using the above formula for Riemannian metric \( g \), it can be easily calculated that

\[
\nabla_{E_1} E_3 = 0, \quad \nabla_{E_2} E_3 = 2E_1, \quad \nabla_{E_3} E_3 = 0, \quad \nabla_{E_3} E_1 = 0, \quad \nabla_{E_1} E_2 = \frac{2}{x} E_1,
\]

\[
\nabla_{E_2} E_1 = -2E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_1} E_1 = -\frac{2}{x} E_2.
\]
From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is a \(N(k)\)-contact metric manifold with \(k = -\frac{4}{x} \neq 0\).

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(E_2, E_3)E_2 = -\frac{4}{x}E_1, \quad R(E_2, E_3)E_1 = \frac{4}{x}E_2,
\]

and the components which can be obtained from these by symmetry property. We shall now show that in such a \(N(k)\)-contact metric manifold the curvature tensor \(R\) is \(\phi\)-recurrent. Since \(\{E_1, E_2, E_3\}\) form a basis of \(M^3\), any vector field \(X \in \chi(M)\) can be taken as

\[
X = a_1E_1 + a_2E_2 + a_3E_3
\]

where \(a_i \in \mathbb{R}^+\) (= the set of all positive real numbers), \(i = 1, 2, 3\). Thus the covariant derivatives of the curvature tensor are given by

\[
(\nabla_X R)(E_2, E_3)E_1 = -\frac{8a_2}{x^2}E_2,
\]

\[
(\nabla_X R)(E_2, E_3)E_2 = \frac{8a_2}{x^2}E_1.
\]

Let us now consider the non-vanishing 1-form \(A(X) = \frac{2a_2}{x}\), at any point \(p \in M\). In our \(M^3\), (2.1) reduces with the 1-form to the following equations:

\[
\phi^2((\nabla_X R)(E_2, E_3)E_1) = A(X)R(E_2, E_3)E_1,
\]

\[
\phi^2((\nabla_X R)(E_2, E_3)E_2) = A(X)R(E_2, E_3)E_2.
\]

This implies that the manifold under consideration is a \(\phi\)-recurrent \(N(k)\)-contact metric manifold, which is neither symmetric nor locally \(\phi\)-symmetric. So, we can state the following:

**Theorem 5.1.** There exists a \(\phi\)-recurrent \(N(k)\)-contact metric manifold, which is neither symmetric nor locally \(\phi\)-symmetric.

**References**


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