Inverse Limits of Spaces with the Weak B-Property

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INVERSE LIMITS OF SPACES WITH THE WEAK \( \mathcal{B} \)-PROPERTY

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Abstract. In this paper we show the following facts. Let \( X \) be the inverse limit space of an inverse system \( \{ X_\alpha, \pi_\beta^\alpha, \Lambda \} \), where \( \Lambda \) is a directed set and \( |\Lambda| = \kappa \geq \omega \). Suppose that each projection \( \pi_\alpha : X \to X_\alpha \) is pseudo-open and \( X \) is \( \kappa \)-paracompact. If each \( X_\alpha \) has the weak \( \mathcal{B} \)-property, then \( X \) has the weak \( \mathcal{B} \)-property. We also show that the analogous result holds for hereditarily weak \( \mathcal{B} \)-property.

Recently, on the studies of products of normality or other covering properties, a series of papers related to the following question proposed (see [1, 2, 3, 9, 10]).

Question. For an inverse system \( \{ X_\alpha, \pi_\beta^\alpha, \Lambda \} \) with \( |\Lambda| = \kappa \geq \omega \), suppose that the inverse limit space \( X \) of \( \{ X_\alpha, \pi_\beta^\alpha, \Lambda \} \) is \( \kappa \)-paracompact and each projection \( \pi_\alpha : X \to X_\alpha \) is pseudo-open. If each \( X_\alpha \) satisfies property \( \mathcal{P} \), then what property \( \mathcal{P} \) can be preserved by the inverse limit space \( X \)?

On the above question, the major work is by Chiba who shows that many covering properties and some separation properties are preserved by the inverse limit space (see [1, 2, 3, 10]). The aim of this paper is to show that the above also holds under the weak \( \mathcal{B} \)-property. About the property \( \mathcal{B} \), the similar results are announced in [1].

The weak \( \mathcal{B} \)-property as a generalization of the property \( \mathcal{B} \) was first introduced by Yasui [7] who gave results relating to the property \( \mathcal{B} \) and countable paracompactness. Subsequently many authors studied this property (see [6]) and in 1985 Rudin [11] renamed this property \( \mathcal{D} \). However, in this paper we shall continue to use the term ”weak \( \mathcal{B} \)-property”.

Throughout this paper, we assume that all spaces are topological spaces without any separation axiom and all maps are continuous.

Let \( X \) be a space and \( A \subset G \subset X \), then \( \overline{A} \), int\( A \) denote the closure, interior of \( A \) in \( X \) respectively and \( \overline{A}^G \), int\( G \)(\( A \)) denote the closure, interior of \( A \) in \( G \) respectively. \( \omega \) denotes the first infinite cardinal and \( [\Sigma]^{<\omega} \) denotes the collection of all non-empty finite subsets of non-empty set \( \Sigma \).

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A map $f$ from $X$ onto $Y$ is called \textit{pseudo-open} if $y \in \text{int} f(U)$ holds for each $y \in Y$ and each open set $U$ in $X$ with $f^{-1}(y) \subset U$.

It is easy to see that both open onto maps and closed onto maps are pseudo-open.

\textbf{Definition 1.} Let $\kappa$ be an arbitrary infinite cardinal. Then the space $X$ is called $\kappa$-\textit{paracompact} if every open cover $\mathcal{U}$ of $X$ with $|\mathcal{U}| \leq \kappa$ has a locally finite open refinement.

The following lemmas are necessary to prove our theorems.

\textbf{Lemma 1 } ([9], Lemma 1.3). A space $X$ is $\kappa$-paracompact if, and only if, for every open cover $\mathcal{U} = \{U_\alpha| \alpha \in \Lambda\}$ of $X$ with $|\Lambda| = \kappa$, there is a locally finite open cover $\mathcal{V} = \{V_\alpha| \alpha \in \Lambda\}$ of $X$ such that $V_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

\textbf{Lemma 2 } ([2], Lemma 2). Let $X$ be a $\kappa$-paracompact space, $\Lambda$ a directed set with $|\Lambda| = \kappa$ and $\mathcal{U} = \{U_\alpha| \alpha \in \Lambda\}$ an open cover of $X$ such that $U_\alpha \subset U_\beta$ for each $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$. Then there is an open cover $\mathcal{V} = \{V_\alpha| \alpha \in \Lambda\}$ of $X$ such that (i) $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$ and (ii) $V_\alpha \subset V_\beta$ if $\alpha \leq \beta$.

Let $X$ be a $\kappa$-paracompact space and $\mathcal{U} = \{U_\alpha| \alpha \in \Lambda\}$ be an open cover of $X$ satisfying the conditions of Lemma 2. Then $\mathcal{U}$ has an open refinement $\mathcal{V} = \{V_\alpha| \alpha \in \Lambda\}$ such that $\overline{V_\alpha} \subset U_\alpha$ (\(\alpha \in \Lambda\)). By Lemma 1, $\mathcal{V}$ has a locally finite open refinement $\{W_\alpha| \alpha \in \Lambda\}$ such that $W_\alpha \subset V_\alpha$ (\(\alpha \in \Lambda\)). Therefore we have the following Lemma.

\textbf{Lemma 3}. Let $X$ be a $\kappa$-paracompact space, $\Lambda$ a directed set with $|\Lambda| = \kappa$ and $\mathcal{U} = \{U_\alpha| \alpha \in \Lambda\}$ an open cover of $X$ such that $U_\alpha \subset U_\beta$ for each $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$. Then there is a locally finite open cover $\mathcal{V} = \{V_\alpha| \alpha \in \Lambda\}$ of $X$ such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$.

\textbf{Definition 2.} [10] Let $\kappa$ be an arbitrary infinite cardinal. A space $X$ has the property $B^*(\kappa)$ if for any decreasing collection $\{F_\alpha| \alpha < \kappa\}$ of closed subsets in $X$ with $\cap\{F_\alpha| \alpha < \kappa\} = \emptyset$, there exists a collection $\{G_\alpha| \alpha < \kappa\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha < \kappa$ and $\cap\{\overline{G_\alpha}| \alpha < \kappa\} = \emptyset$.

A space $X$ has the \textit{weak $B$-property} if $X$ has the property $B^*(\kappa)$ for every infinite cardinal $\kappa$.

It is easy to see that:

\textbf{Proposition 1}. For a space $X$, the following are equivalent:

\item Proposition 2

\item Proposition 3
(1) $X$ has the property $B^*(\kappa)$.

(2) [5] Every increasing open cover $\{U_{\alpha}\mid \alpha < \kappa\}$ of $X$ has an open cover $\{V_{\alpha}\mid \alpha < \kappa\}$ of $X$ such that $\overline{V_{\alpha}} \subset U_{\alpha}$ for each $\alpha < \kappa$.

Theorem 1. Let $\{X_{\alpha}, \pi_{\beta}^\alpha, \Lambda\}$ be an inverse system and $X$ be its inverse limit space $\lim \{X_{\alpha}, \pi_{\beta}^\alpha, \Lambda\}$. Suppose that each projection $\pi_{\alpha}: X \to X_{\alpha}$ is a pseudo-open map and $X$ is a $\kappa$-paracompact space, where $|\Lambda| = \kappa$. If each $X_{\alpha}$ has the weak $B$-property, then $X$ also has the weak $B$-property.

Proof. Let $\tau$ be an arbitrary infinite cardinal and $G = \{G_{\xi}\mid \xi < \tau\}$ be an increasing open cover of $X$. For each $\alpha \in \Lambda$ and $\xi < \tau$, we put $U_{\alpha, \xi} = \bigcup \{U\mid U \text{ open in } X_{\alpha}, \pi_{\alpha}^{-1}(U) \subset G_{\xi}\}$ and $U_{\alpha} = \bigcup \{U_{\alpha, \xi}\mid \xi < \tau\}$, then the collection $\{\pi_{\alpha}^{-1}(U_{\alpha})\mid \alpha \in \Lambda\}$ satisfies:

1. $\bigcup \{\pi_{\alpha}^{-1}(U_{\alpha})\mid \alpha \in \Lambda\} = X$.

2. $\pi_{\alpha}^{-1}(U_{\alpha}) \subset \pi_{\beta}^{-1}(U_{\beta})$ if $\alpha \leq \beta$.

Since $X$ is $\kappa$-paracompact, by Lemma 2, there is an open cover $\{W_{\alpha}\mid \alpha \in \Lambda\}$, such that

3. $W_{\alpha} \subset \pi_{\alpha}^{-1}(U_{\alpha})$ for each $\alpha \in \Lambda$.

4. If $\alpha \leq \beta$, then $W_{\alpha} \subset W_{\beta}$.

Now, for each $\alpha \in \Lambda$, we define the closed subset $T_{\alpha} = X_{\alpha} \setminus \text{int}_{\alpha}(X \setminus \overline{W_{\alpha}})$ of $X_{\alpha}$. Since each projection $\pi_{\alpha}: X \to X_{\alpha}$ is a pseudo-open map, we have

5. $T_{\alpha} \subset U_{\alpha}$ for each $\alpha \in \Lambda$.

Put $C_{\alpha} = \text{int}_{\alpha}(T_{\alpha})$ for each $\alpha \in \Lambda$, then

6. $\{C_{\alpha}\mid \alpha \in \Lambda\}$ is an open cover of $X$.

Because, for each $x \in X$, some $W_{\alpha}$ contains $x$. Hence there are $\beta \in \Lambda$ and an open subset $V$ in $X_{\beta}$ such that $x \in \pi_{\beta}^{-1}(V) \subset W_{\alpha}$. Then there is $\gamma \in \Lambda$ with $\alpha, \beta \leq \gamma$, and $x \in \pi_{\beta}^{-1}(V) \subset \pi_{\gamma}^{-1}(T_{\gamma})$. Hence $x \in C_{\gamma}$.

Since $X$ is $\kappa$-paracompact, by Lemma 1, there is a locally finite open cover $\{O_{\alpha}\mid \alpha \in \Lambda\}$ of $X$ such that $O_{\alpha} \subset C_{\alpha}$ for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$ and $\xi < \tau$, we put $U'_{\alpha, \xi} = U_{\alpha, \xi} \cup (X_{\alpha} \setminus T_{\alpha})$, then $\{U'_{\alpha, \xi}\mid \xi < \tau\}$ is an increasing open cover of $X_{\alpha}$ since $\{U_{\alpha, \xi}\mid \xi < \tau\}$ is an increasing open cover of $U_{\alpha}$. Thus we have

7. $\{X_{\alpha} \setminus U'_{\alpha, \xi}\mid \xi < \tau\}$ is a decreasing collection of closed subsets of $X_{\alpha}$ satisfying that $\bigcap \{X_{\alpha} \setminus U'_{\alpha, \xi}\mid \xi < \tau\} = \emptyset$.

Since $X_{\alpha}$ has the weak $B$-property, there exists an open collection $\{V_{\alpha, \xi}\mid \xi < \tau\}$ of $X_{\alpha}$ such that

8. $X_{\alpha} \setminus U'_{\alpha, \xi} \subset V_{\alpha, \xi}$ for each $\xi < \tau$.

9. $\bigcap \overline{V_{\alpha, \xi}} = \emptyset$. 

To show that $X$ has the weak $B$-property, by Proposition 1, it is sufficient to construct an open cover $\{A_\xi|\xi<\tau\}$ of $X$ satisfying $A_\xi \subset G_\xi$ for each $\xi<\tau$. Let $A_\xi = \cup \{\pi_\alpha^{-1}(X_\alpha \setminus V_{\alpha,\xi}) \cap O_\alpha|\alpha \in \Lambda\}$ for each $\xi<\tau$. Then (10) $\{A_\xi|\xi<\tau\}$ is an open cover of $X$.

In the fact, for each $x \in X$, $x \in O_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since $\pi_{\alpha_0}(x) \in X_{\alpha_0}$ and $\cap \{V_{\alpha_0,\xi}|\xi<\tau\} = \emptyset$, we have $x_{\alpha_0} \in X_{\alpha_0} \setminus V_{\alpha_0,\xi_0}$ for some $\xi_0 < \tau$. Hence $x \in \pi_{\alpha_0}^{-1}(X_{\alpha_0} \setminus V_{\alpha_0,\xi_0}) \cap O_{\alpha_0} \subset A_{\xi_0}$.

Last, we show that $A_\xi \subset G_\xi$ for each $\xi<\tau$. In the fact, for each $\xi<\tau$, observe that the collection $\{\pi_\alpha^{-1}(X_\alpha \setminus V_{\alpha,\xi}) \cap O_\alpha|\alpha \in \Lambda\}$ is locally finite in $X$. Therefore for each $\xi<\tau$,

$$
A_\xi = \cup \{\pi_\alpha^{-1}(X_\alpha \setminus V_{\alpha,\xi}) \cap O_\alpha|\alpha \in \Lambda\} \subset \cup \{\pi_\alpha^{-1}(X_\alpha \setminus V_{\alpha,\xi}) \cap O_\alpha|\alpha \in \Lambda\}
$$

$$
\subset \cup \{\pi_\alpha^{-1}(U_{\alpha,\xi}) \cap \pi_\alpha^{-1}(T_\alpha)|\alpha \in \Lambda\}
$$

$$
= \cup \{\pi_\alpha^{-1}((U_{\alpha,\xi} \cup (X_\alpha \setminus T_\alpha)) \cap T_\alpha)|\alpha \in \Lambda\}
$$

$$
= \cup \{\pi_\alpha^{-1}(U_{\alpha,\xi} \cap T_\alpha)|\alpha \in \Lambda\} \subset \cup \{\pi_\alpha^{-1}(U_{\alpha,\xi})|\alpha \in \Lambda\} \subset G_\xi
$$

The proof of Theorem 1 is completed.

We describe inverse limit spaces of the hereditarily weak $B$-properties.

A space $X$ has the hereditarily weak $B$-property if every subspace of $X$ has the weak $B$-property.

It is not difficult to show the following lemma.

**Proposition 2.** A space $X$ has the hereditarily weak $B$-property i, and only if, every open subspace of $X$ has the weak $B$-property.

**Theorem 2.** Let $\{X_\alpha, \pi_\alpha^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\alpha^\alpha, \Lambda\}$. Suppose that $G$ is a $\kappa$-paracompact open subspace of $X$. If each $X_\alpha$ has the hereditarily weak $B$-property, then $G$ has the weak $B$-property.

**Proof.** Let $\tau$ be an arbitrary infinite cardinal and $G = \{G_\xi|\xi<\tau\}$ be an increasing open cover of $G$. For each $\alpha \in \Lambda$ and $\xi<\tau$, we put $U_{\alpha,\xi} = \cup \{U|U$ open in $X_\alpha, \pi_\alpha^{-1}(U) \subset G_\xi\}$ and $U_\alpha = \cup \{U_{\alpha,\xi}|\xi<\tau\}$, then similar to the case of Theorem 1, the collection $\{\pi_\alpha^{-1}(U_\alpha)|\alpha \in \Lambda\}$ satisfies

1. $\cup \{\pi_\alpha^{-1}(U_\alpha)|\alpha \in \Lambda\} = G$ and
2. $\pi_\alpha^{-1}(U_\alpha) \subset \pi_\beta^{-1}(U_\beta)$ if $\alpha \leq \beta$.

Since $G$ is a $\kappa$-paracompact open subspace of $X$ and $\{\pi_\alpha^{-1}(U_\alpha)|\alpha \in \Lambda\}$ satisfies the condition of Lemma 3, there is a locally finite open refinement.
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\{O_\alpha | \alpha \in \Lambda\} of \\{\pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\} in \ G such that \overline{O_\alpha \cap U_\alpha} \subset \pi_\alpha^{-1}(U_\alpha) for each \ \alpha \in \Lambda.

Note that the collection \(U_\alpha = \{U_{\alpha, \xi} | \xi < \tau\}\) is an increasing open cover of \(U_\alpha\) because \(G\) is an increasing open cover of \(G\). Therefore \(\{U_\alpha \setminus U_{\alpha, \xi} | \xi < \tau\}\) is a decreasing closed collection of \(U_\alpha\) satisfying that \(\cap U_{\alpha, \xi} = \emptyset\).

Since \(U_\alpha\) has the weak \(B\)-property, there exists an open collection \(\{V_{\alpha, \xi} | \xi < \tau\}\) in \(U_\alpha\) such that

1. \(U_\alpha \setminus U_{\alpha, \xi} \subset V_{\alpha, \xi}\) for each \(\xi < \tau\) and
2. \(\cap \{V_{\alpha, \xi} \setminus U_{\alpha, \xi} | \xi < \tau\} = \emptyset\).

To show that \(G\) has the weak \(B\)-property, by Proposition 1, it is sufficient to construct an open cover \(\{A_\xi | \xi < \tau\}\) of \(G\) satisfying \(\overline{A_\xi \cap G} \subset G_\xi\) for each \(\xi < \tau\).

Now, for each \(\xi < \tau\), we put \(A_\xi = \cup \{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap O_\alpha | \alpha \in \Lambda\}\). Then similar to the proof of Theorem 1, \(\{A_\xi | \xi < \tau\}\) is an open cover of \(G\). Moreover we have

1. \(\overline{A_\xi \cap G} \subset G_\xi\) for each \(\xi < \tau\)

Indeed, for each \(\xi < \tau\), since the collection \(\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap O_\alpha | \alpha \in \Lambda\}\) is locally finite in \(G\), we have

\[
\overline{A_\xi} = \overline{\cup \left\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap O_\alpha \right\} | \alpha \in \Lambda}
\]
\[
\subset \overline{\cup \left\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap O_\alpha \right\} | \alpha \in \Lambda}
\]
\[
\subset \overline{\cup \left\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap G \cap \pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\right\}}
\]
\[
\subset \overline{\cup \left\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap G \cap \pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\right\}}
\]
\[
= \overline{\cup \left\{\pi_\alpha^{-1}(U_\alpha \setminus V_{\alpha, \xi}) \cap G | \alpha \in \Lambda\right\}} \subset \overline{\cup \left\{\pi_\alpha^{-1}(U_{\alpha, \xi} \cap G) | \alpha \in \Lambda\right\}} \subset G_\xi
\]

The proof of Theorem 2 is completed.

By Proposition 2 and Theorem 2, we obtain the following usual statement of inverse limits of the spaces with the hereditarily weak \(B\)-property.

**Corollary 1.** Let \(\{X_\alpha, \pi_\alpha^{\beta}, \Lambda\}\) be an inverse system and \(X = \lim \{X_\alpha, \pi_\alpha^{\beta}, \Lambda\}\). Suppose that \(X\) is a hereditarily \(\kappa\)-paracompact space, where \(|\Lambda| = \kappa\). If each \(X_\alpha\) has the hereditarily weak \(B\)-property, then \(X\) also has the hereditarily weak \(B\)-property.

Finally, we study properties of the product of spaces with the weak \(B\)-properties.
Let $\kappa$ be an infinite cardinal number, $\{X_\alpha | \alpha \in \Sigma\}$ be a collection of spaces with $|\Sigma| = \kappa$ and $X = \prod_{\alpha \in \Sigma} X_\alpha$. We define the relation ”$\leq$” of $[\Sigma]^{<\omega}$ as $A \leq B$ if and only if $A \subset B$ for $A, B \in [\Sigma]^{<\omega}$ and define the finite subproduct $Z_A = \prod_{\alpha \in A} X_\alpha$ of $X$ for every $A \in [\Sigma]^{<\omega}$. Then $[\Sigma]^{<\omega}$ is a directed set with the relation ”$\leq$”. For $A \leq B$ ($A, B \in [\Sigma]^{<\omega}$), let $\pi_B^A : Z_B \rightarrow Z_A$ be the natural projection. Then $\pi_B^A$ is an open bonding map from $Z_B$ onto $Z_A$ and hence we obtain the inverse system $Z_A$ induced by the collection $\{X_\alpha | \alpha \in \Sigma\}$, and by Lemma 1 of [2], every projection $\pi_A$ of this inverse system $\{Z_A, \pi_B^A, [\Sigma]^{<\omega}\}$ is also an open map from the inverse limit space $Z = \lim\downarrow \{Z_A, \pi_B^A, [\Sigma]^{<\omega}\}$ onto $Z_A$.

**Lemma 4.** [8] Suppose that $\{X_\alpha | \alpha \in \Sigma\}$ is a collection of spaces, $X = \prod_{\alpha \in \Sigma} X_\alpha$ and $Z = \lim\downarrow \{Z_A, \pi_B^A, [\Sigma]^{<\omega}\}$, where $|\Sigma| \geq \omega$. Then $X$ and $Z$ are homeomorphic.

**Theorem 3.** Suppose that $\{X_\alpha | \alpha \in \Sigma\}$ is a collection of Hausdorff spaces and its product space $X = \prod_{\alpha \in \Sigma} X_\alpha$ is $\kappa$-paracompact, where $|\Sigma| = \kappa \geq \omega$. Then $X$ has the weak $\mathcal{B}$-property if, and only if, for each $A \in [\Sigma]^{<\omega}$ the finite subproduct $Z_A = \prod_{\alpha \in A} X_\alpha$ has the weak $\mathcal{B}$-property.

**Proof.** The ”only if” part follows from Lemma 4 and Theorem 1. For the ”if” part, assume that $X$ has the weak $\mathcal{B}$-property. Then for each $A \in [\Sigma]^{<\omega}$, we choose a fixed point $x_\alpha \in X_\alpha$ for every $\alpha \in \Sigma \setminus A$. Since $Z_A$ is homeomorphic to the closed subspace $\prod_{\alpha \in A} X_\alpha \times \prod_{\alpha \in \Sigma \setminus A} \{x_\alpha\}$ of $X$, $Z_A$ has the weak $\mathcal{B}$-property.

The proof of Theorem 3 is completed.

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