Geometry on Grassmann Manifolds G(2,8) and G(3,8)

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Abstract

In this paper, we use the Clifford algebra $\mathbb{C}_\ell^8$ to construct fibre bundles $\hat{\iota}_1: G(2; 8) \to S^6$, $\hat{\iota}'_1: G(2; 7) \to S^6$ and $\hat{\iota}_2: G(3; 8) \to S^7$, the fibres are $\mathbb{C}P^3$, $\mathbb{C}P^2$ and AOC $= G_2 = SO(4)$ respectively. We show that $G(2; 5)$, $\mathbb{C}P^3$ and $S^6$ are the homologically volume minimizing submanifolds of $G(2; 8)$ by calibrations and they generate the homology group $H_6(G(2; 8))$. The submanifolds $S^7$ and AOC of $G(3; 8)$ generate $H_7(G(3; 8))$ and $H_8(G(3; 8))$ respectively.

KEYWORDS: Grassmann manifold, Riemann connection, Clifford algebra, fibre bundle, calibration.

GEOMETRY ON GRASSMANN MANIFOLDS
\( G(2,8) \) AND \( G(3,8) \)

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Abstract. In this paper, we use the Clifford algebra \( C_\ell^8 \) to construct fibre bundles \( \tau_1: G(2,8) \to S^6 \), \( \tau'_1: G(2,7) \to S^6 \) and \( \tau_2: G(3,8) \to S^7 \), the fibres are \( CP^3 \), \( CP^2 \) and \( ASSOC = G_2/SO(4) \) respectively. We show that \( G(2,5) \), \( CP^3 \) and \( S^6 \) are the homologically volume minimizing submanifolds of \( G(2,8) \) by calibrations and they generate the homology group \( H_6(G(2,8)) \). The submanifolds \( S^7 \) and \( ASSOC \) of \( G(3,8) \) generate \( H_7(G(3,8)) \) and \( H_8(G(3,8)) \) respectively.

\[ \mathbb{1}. \text{Introduction} \]

As is well-known, the Grassmann manifold \( G(2,4) \) is a fibre bundle over \( S^2 \). In this paper, we use the Clifford algebra \( C_\ell^8 \) to define maps \( \tau_1: G(2,8) \to S^6 \), \( \tau'_1: G(2,7) \to S^6 \) and \( \tau_2: G(3,8) \to S^7 \), which make the Grassmann manifolds \( G(2,8) \), \( G(2,7) \) and \( G(3,8) \) fibre bundles. The fibres are complex projective spaces \( CP^3 \), \( CP^2 \) and \( ASSOC = G_2/SO(4) \) respectively. The fibres of these bundles are also the totally geodesic submanifolds of \( G(2,8) \), \( G(2,7) \) and \( G(3,8) \) respectively.

By calibrations, we show that the submanifolds \( G(2,5) \), \( CP^3 \) and \( S^6 \) of \( G(2,8) \) are the volume-minimizing cycles in \( G(2,8) \) and they generate the homology group \( H_6(G(2,8)) \). These gives an answer to the problem (5) in [3]. The submanifolds \( S^7 \) and \( ASSOC \) are also the generators of \( H_7(G(3,8)) \), \( H_8(G(3,8)) \) respectively.

In this paper, we also show that the Stiefel manifold \( V_8,2 \) is homeomorphic to the product of two spheres \( S^7 \times S^6 \).
Define the subspace $V = V^+ \oplus V^-$ of $C\ell_8$ by $V^+ = C\ell_8^{\text{even}}$ and $V^- = C\ell_8^{\text{odd}}$, where

$$A = \text{Re}[(\bar{e}_1 + \sqrt{-1}\bar{e}_2) \cdots (\bar{e}_7 + \sqrt{-1}\bar{e}_8)(1 + \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7)].$$

**Lemma 1.** The space $V = V^+ \oplus V^-$ is an irreducible module over $C\ell_8$. The spaces $V^+$ and $V^-$ are generated by $\bar{e}_1\bar{e}_B A$ and $\bar{e}_B A$ ($B = 1, \ldots, 8$) respectively.

**Proof.** For proof see [7], [9]. In the following we give another proof. By a computation, we have

$$A = \bar{e}_1\bar{e}_3\bar{e}_5\bar{e}_7 + \bar{e}_2\bar{e}_4\bar{e}_6\bar{e}_8 - \bar{e}_1\bar{e}_3\bar{e}_6\bar{e}_8 - \bar{e}_2\bar{e}_4\bar{e}_5\bar{e}_7$$

$$- \bar{e}_1\bar{e}_4\bar{e}_5\bar{e}_8 - \bar{e}_1\bar{e}_4\bar{e}_6\bar{e}_7 - \bar{e}_2\bar{e}_3\bar{e}_5\bar{e}_8 - \bar{e}_2\bar{e}_3\bar{e}_6\bar{e}_7$$

$$+ 1 + \bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4\bar{e}_5\bar{e}_6\bar{e}_7\bar{e}_8 - \bar{e}_5\bar{e}_6\bar{e}_7\bar{e}_8 - \bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4$$

$$- \bar{e}_3\bar{e}_4\bar{e}_7\bar{e}_8 - \bar{e}_1\bar{e}_2\bar{e}_5\bar{e}_6 - \bar{e}_1\bar{e}_2\bar{e}_7\bar{e}_8 - \bar{e}_3\bar{e}_4\bar{e}_5\bar{e}_6.$$

It is easy to see that for any $1 \leq i_1 < i_2 < i_3 \leq 8$, there is a term in $A - 1 - \bar{e}_1 \cdots \bar{e}_8$ which contains $\bar{e}_{i_1} \bar{e}_{i_2} \bar{e}_{i_3}$. Furthermore, $A$ is invariant by acting every summand of $A$ on itself, then $A \cdot A = 16A$. These shows $V^+$ and $V^-$ are generated by $\bar{e}_1\bar{e}_B A$ and $\bar{e}_B A$ ($B = 1, \ldots, 8$) respectively. For the dimensional reason, $V$ is an irreducible module over $C\ell_8$. These generators of $V$ can be used to construct the isomorphism between the Clifford algebra $C\ell_8$ and the matrix algebra $R(16)$.

Let $G(k, 8)$ be the Grassmann manifold formed by all oriented $k$-dimensional subspaces of $R^8$. For any $x \in G(k, 8)$, there are orthonormal vectors $e_1, \ldots, e_k$ such that $x$ can be represented by $e_1 \wedge \cdots \wedge e_k$. Thus $G(k, 8)$ becomes a submanifold of the space $\bigwedge^k (R^8)$. The spaces $\bigwedge(R^8)$ and $C\ell_8$ are isomorphic as a vector space. Identify the elements $e_1 \wedge \cdots \wedge e_k$ with $e_1 \cdots e_k$, $G(k, 8)$ can also be viewed as a subset of the Clifford algebra $C\ell_8$. Then for any $x \in G(k, 8)$, there is $v \in R^8$ such that $xA = \bar{e}_1 vA$ or $xA = vA$ according to the number $k$ being even or odd. With the inner product defined on $C\ell_8$ naturally, we can show $|v| = 1$. Thus we have a map $G(k, 8) \to S^7$, $x \mapsto v$. It is not difficult to see that if $k = 4$, the map $G(4, 8) \to S^7$ can not be a fibre bundle: the dimensions of the fibres over $\pm \bar{e}_1 \in S^7$ are different from that of the other fibres. Since the Grassmann manifolds $G(2, 8)$ and $G(6, 8)$, $G(3, 8)$ and $G(5, 8)$ are isometric respectively, we need only to study $G(2, 8)$ and $G(3, 8)$.

Let $e_1, e_2, \ldots, e_8$ be an orthonormal frame fields on $R^8$ such that $e_1 \wedge \cdots \wedge e_k$ generate a neighborhood of $x$ in $G(k, 8)$. By

$$d(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} \sum_{\alpha=k+1}^{8} \omega_i^\alpha E_{i\alpha}, \quad \omega_i^\alpha = \langle de_i, e_\alpha \rangle,$$
we know that the elements \( E_{i\alpha} = e_1 \ldots e_{i-1} e_\alpha e_{i+1} \ldots e_k \) \((i = 1, \ldots, k, \alpha = k+1, \ldots, 8)\) can be looked as a basis of \( T^* e_1 \ldots e_k G(k, 8) \), and \( \omega^\alpha_1 \) are its dual. The metric on \( G(k, 8) \) is \( ds^2 = \sum_{i=1}^{k} \sum_{\alpha=k+1}^{8} (\omega^\alpha_i)^2 \). Differentiate \( E_{i\alpha} \) we get the Riemannian connection \( \nabla \) on \( G(k, 8) \),

\[
\nabla E_{i\alpha} = \sum_{j=1}^{k} \omega^j_i E_{j\alpha} + \sum_{\beta=k+1}^{8} \omega^\beta_i E_{i\beta}.
\]

Bryant [1] has shown that the Lie group \( Spin_7 \) is the isotropy group of \( SO(8) \) acting on \( A \), that is, \( Spin_7 = \{ G \in SO(8) \mid G(A) = A \} \). He also shows that \( Spin_7 \) acts on \( G(2, 8) \), \( G(3, 8) \) and \( S^7 \) transitively. The subgroup \( G_2 = \{ G \in Spin_7 \mid G(\bar{e}_1) = \bar{e}_1 \} \) acts transitively on \( S^6 = \{ v \in S^7 \mid v \perp \bar{e}_1 \} \).

**Theorem 2.** There is a map \( \tau_1 : G(2, 8) \to S^6 \) which makes \( G(2, 8) \) a fibre bundle. The fibres are diffeomorphic to the complex projective space \( CP^3 \).

**Proof.** First we show that if \( xA = \bar{e}_1 vA, x \in G(2, 8) \), then \( v \perp \bar{e}_1 \). Denote \( \langle \ , \ , \ \rangle \) the inner product on \( C\ell_8 \). Let \( v = a\bar{e}_1 + bv', v' \perp \bar{e}_1 \). By \( A \cdot A = 16A, x \cdot x = -1, \langle xA, A \rangle = 16 \langle x, A \rangle = 0 \) and

\[
16 = \langle xA, \bar{e}_1 vA \rangle = \langle xA, b\bar{e}_1 v' A \rangle \leq 16|b|,
\]

we have \( v \perp \bar{e}_1 \). Hence we have a map \( \tau_1 : G(2, 8) \to S^6, \tau_1(x) = v \).

For any \( G \in Spin_7 \), we have the following commutative diagram

\[
\begin{array}{ccc}
G(2, 8) & \xrightarrow{G} & G(2, 8) \\
\downarrow{\tau_1} & & \downarrow{\tau_1} \\
S^6 & \xrightarrow{G} & S^6,
\end{array}
\]

where \( G \) is defined by \( G(xA) = \bar{e}_1 G(vA), v = \tau_1(x) \). Thus the fibres of \( \tau_1 \) are all diffeomorphic. Let \( J_0 \) be the complex structure on \( R^8 \): \( J_0 \bar{e}_{2s-1} = \bar{e}_{2s} \) \((s = 1, \ldots, 4)\). As \( A \cdot A = 16A \), we can show that \( xA = \bar{e}_1 \bar{e}_2 A \) if and only if \( \langle xA, \bar{e}_1 \bar{e}_2 A \rangle = 1 \). For any \( x \in G(2, 8) \), we can write \( x = v \wedge (aJ_0 v + w), w \perp v, J_0 v \). Since the two form part of \( \bar{e}_1 \bar{e}_2 A \) is invariant under the action of the unitary group \( U(n) \), we can show \( vJ_0 vA = \bar{e}_1 \bar{e}_2 A \) and \( \langle vwA, \bar{e}_1 \bar{e}_2 A \rangle = 0 \). Then we have

\[
\tau_1^{-1}(\bar{e}_2) = \{ w \wedge J_0(w) \mid w \in S^7 \} \approx CP^3.
\]
Lemma 3. Let $e_1, e_2, \ldots, e_8$ be $\text{Spin}_7$ frame fields on $R^8$. Then the 1-forms $\omega_B^C = \langle de_B, e_C \rangle$ satisfy
\[
\begin{align*}
\omega_1^2 + \omega_3^4 + \omega_5^6 + \omega_7^8 &= 0, \\
\omega_1^4 + \omega_2^3 + \omega_5^8 + \omega_6^7 &= 0, \\
\omega_1^6 + \omega_2^5 - \omega_3^8 - \omega_4^1 &= 0, \\
\omega_1^8 + \omega_2^7 + \omega_3^6 + \omega_4^5 &= 0.
\end{align*}
\]

Proof. As the Lie group $\text{Spin}_7$ is the isotropy group of $A$, the element $A$ can also be represented by
\[
A = \text{Re}[(e_1 + \sqrt{-1}e_2) \cdots (e_7 + \sqrt{-1}e_8)(1 + e_1e_3e_5e_7)].
\]
Since $-e_1e_2e_3e_4A = A$, differentiate it we can get the last four equations of the lemma. The other equations can be proved similarly.

In the following, we study the fibres of $\tau_1$. Let $x \in \tau_1^{-1}(v), v \in S^6$, and $e_1, e_2, \ldots, e_8$ be $\text{Spin}_7$ frame fields on $R^8$ such that the elements $e_1 \wedge e_2$ generate a neighborhood $U \subset \tau_1^{-1}(v)$ of $x$.

From $d(e_1e_2A) = d(e_1e_2A) = d((e_1v)A) = 0$, we have
\[
\omega_2^{s+1} + \omega_1^{2s+2} = 0, \quad \omega_2^{2s+2} - \omega_1^{s+1} = 0 \ (s = 1, 2, 3).
\]
Then on $U$,
\[
d(e_1 \wedge e_2) = \sum_{s=1}^3 [\omega_1^{2s+1} (E_1 2s+1 + E_2 2s+2) + \omega_1^{2s+2} (E_1 2s+2 - E_2 2s+1)].
\]
Hence the vectors $\tilde{E}_{2s+1} = E_1 2s+1 + E_2 2s+2$ and $\tilde{E}_{2s+2} = E_1 2s+2 - E_2 2s+1$ form a basis of $T_{e_1 \wedge e_2} \tau_1^{-1}(v)$, and $\omega_1^\alpha (\alpha = 3, \ldots, 8)$ are its dual.

Lemma 4. The Riemannian connection $D$ on the fibre $\tau_1^{-1}(v)$ is given by
\[
D\tilde{E}_{2s+1} = -\omega_1^{2s+1} \tilde{E}_{2s+2} + \sum_{\beta=3}^8 \omega_2^{2s+1} \tilde{E}_\beta, \quad D\tilde{E}_{2s+2} = \omega_1^{2s+2} \tilde{E}_{2s+1} + \sum_{\beta=3}^8 \omega_2^{2s+2} \tilde{E}_\beta.
\]
The fibre $\tau_1^{-1}(v)$ is a totally geodesic submanifold of $G(2, 8)$.

Proof. Restricting the Riemannian connection of $G(2, 8)$ on $\tau_1^{-1}(v)$, by Lemma 3, we have $\nabla \tilde{E}_\alpha = D\tilde{E}_\alpha$. Then $\tau_1^{-1}(v)$ is a totally geodesic submanifold of $G(2, 8)$.

The action of $\text{Spin}_7$ on $G(2, 8)$ is isometry and preserves the fibres of $\tau_1$. To prove $\tau_1^{-1}(v)$ is a totally geodesic submanifold we need only to show this is true for $v = e_2$ which can be proved directly (without Lemma 3).
Let $G(2, 7)$ be the Grassmann manifold on $R^7 = \{v \in R^8 \mid v \perp \bar{e}_1\}$. Restricting the map $\tau_1$ on $G(2, 7)$, we have map $\tau'_1: G(2, 7) \to S^6$. For any $G \in G_2$, we have the following commutative diagram

$$
\begin{array}{ccc}
G(2, 7) & \xrightarrow{G} & G(2, 7) \\
\downarrow \tau'_1 & & \downarrow \tau'_1 \\
S^6 & \xrightarrow{G} & S^6.
\end{array}
$$

These shows

**Theorem 5.** There is a map $\tau'_1: G(2, 7) \to S^6$ which makes $G(2, 7)$ a fibre bundle. The fibres are homeomorphic to the complex projective space $CP^2$ and are totally geodesic submanifolds of $G(2, 7)$.

The fibre of $\tau'_1$ over $\bar{e}_2$ is $\tau'^{-1}_1(\bar{e}_2) = \{u \wedge J_0u \mid u \in S^5\}$, where $S^5 = \{u \in S^7 \mid u \perp \bar{e}_1, \bar{e}_2\}$. Restricting the map $\tau_1$ on $G(2, 6)$ is not a fibre bundle. We can also show that the map $\tau_1: G(2, 8) \to S^6(\sqrt{2}/2)$ is a Riemannian submersion, but $\tau'_1: G(2, 7) \to S^6(r)$ can not be a Riemannian submersion for any $r > 0$.

Similar to the case of $G(2, 8)$, there is a commutative diagram for each $G \in Spin_7$:

$$
\begin{array}{ccc}
G(3, 8) & \xrightarrow{G} & G(3, 8) \\
\downarrow \tau'_2 & & \downarrow \tau'_2 \\
S^7 & \xrightarrow{G} & S^7.
\end{array}
$$

Denote $\tau'^{-1}_2(\bar{e}_1)$ by $ASSOC$ which is homeomorphic to $G_2/SO(4)$, see [3], [4].

**Theorem 6.** The map $\tau_2: G(3, 8) \to S^7$ is a fibre bundle, the fibre type is $ASSOC$ and the fibres are totally geodesic submanifolds.

The proof of Theorem 6 is similar to that of Theorem 2 and Lemma 4. By Theorem 6 we have the following corollaries:

1. If $v = \tau_2(x)$, we can show $v \perp x$. Then we have another fibre bundle $G(3, 8) \to CAYLEY$, $x \mapsto x \wedge v$, the fibre type is $G(3, 4) \approx S^3$, where $CAYLEY = \{y \in G(4, 8) \mid yA = A\}$ is a totally geodesic submanifolds of $G(4, 8)$;

2. The space $E = \{(x, w) \in G(3, 8) \times R^8 \mid w \perp x, w \perp v = \tau_2(x)\}$ is a vector bundle over $G(3, 8)$ with fibre $R^4$;

3. Combing Hopf bundles $S^7 \to CP^3$ and $S^7 \to HP^1$, we get two fibre bundles $G(3, 8) \to CP^3$ and $G(3, 8) \to S^4$. 


In the following we give more applications of the Clifford algebra $C\ell_8$.

Let $V_{8,2}$ be the Stiefel manifold. We have maps

$$V_{8,2} \xrightarrow{\pi} G(2,8) \xrightarrow{\tau_1} S^6.$$  

For any $v \in S^6$, define a linear map $J_v: R^8 \to R^8$ by $J_v(\bar{e}_1) = v$, $J_v(v) = -\bar{e}_1$, and for any $w \perp \bar{e}_1, v$, $J_v(w)$ is defined by $J_v(w)A = -\bar{e}_1vwA$. It is not difficult to see that $J_v$ is a complex structure on $R^8$ and $J_{\bar{e}_2} = J_0$ as defined in the proof of Theorem 2. It is easy to show that $J_v = G^{-1}J_0G$ for any $G \in G_2$ such that $G(v) = \bar{e}_2$. Then the fibre of $\tau_1$ over $v \in S^6$ can be represented by $\tau_1^{-1}(v) = \{u \wedge J_vu \mid u \in S^7\}$. This shows (see also [6], p. 37)

$$V_{8,2} = \{(u, J_vu) \mid u \in S^7, v \in S^6\} \approx S^7 \times S^6.$$  

Similarly, for the Stiefel manifold $V_{7,2}$, there are maps

$$V_{7,2} \xrightarrow{\pi'} G(2,7) \xrightarrow{\tau'_1} S^6.$$  

We can show that

$$V_{7,2} = \{(u, J_vu) \mid u, v \in S^6, u \perp v\} \subset S^6 \times S^6.$$  

Now we give some representation of Hopf maps. Let $w_1 = \sum_{i=1}^4 v_i\bar{e}_i$, $w_2 = \sum_{j=5}^8 v_j\bar{e}_j$ with $\sum_{i=1}^8 v_i^2 = 1$. By $w_sJ_0w_sA = |w_s|^2\bar{e}_1\bar{e}_2A$ ($s = 1, 2$) and $w_1J_0w_2A = -w_2J_0w_1A$, we have

$$(w_1 + w_2)(J_0w_1 - J_0w_2)A = (|w_1|^2 - |w_2|^2)\bar{e}_1\bar{e}_2A + 2w_2J_0w_1A.$$  

Computing the right hand side of this equation, we find a map $\eta: S^7 \to S^4$,

\[
\begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_8
\end{pmatrix} \mapsto \begin{pmatrix}
  |w_1|^2 - |w_2|^2 \\
  2(v_1v_6 + v_2v_5 + v_3v_8 + v_4v_7) \\
  2(v_1v_5 - v_2v_6 - v_3v_7 + v_4v_8) \\
  2(v_1v_8 + v_2v_7 - v_3v_6 - v_4v_5) \\
  2(v_1v_7 - v_2v_8 + v_3v_5 - v_4v_6)
\end{pmatrix}.
\]

Let $z_1 = v_1 + iv_2 + jv_3 + kv_4$, $z_2 = v_6 + iv_5 + jv_8 + kv_7$, where $i, j, k$ are the quaternion numbers with $k = ij$. The map $\eta$ can also be obtained from the map

$$(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\bar{z}_1 \cdot z_2).$$  

It is easy to see that $\eta$ is the Hopf map $S^7 \to HP^1$. 

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From
\[
\left(\sum_{i=1}^{4} v_i \tilde{e}_i \sum_{i=1}^{4} v_i \tilde{e}_{i+4}\right) A
= \tilde{e}_1[(v_1^2 - v_2^2 + v_3^2 - v_4^2)\tilde{e}_5 + 2(v_1 v_2 - v_3 v_4)\tilde{e}_6 + 2(v_1 v_4 + v_2 v_3)\tilde{e}_8] A,
\]
we have the Hopf map \(S^3 \to S^2 = CP^1\),
\[
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \mapsto \begin{pmatrix} v_1^2 - v_2^2 + v_3^2 - v_4^2 \\ 2v_1 v_2 - 2v_3 v_4 \\ 2v_1 v_4 + 2v_2 v_3 \end{pmatrix}.
\]

§3. The Calibrations on \(G(2, 8)\) and \(G(3, 8)\)

Let \(\xi\) be a closed \(k\)-form on a Riemannian manifold \(M\). If \(\xi(X_1, \ldots, X_k) \leq 1\) for any \(p \in M\) and for any orthonormal vectors \(X_1, \ldots, X_k \in T_p M\), we call \(\xi\) a calibration on \(M\), see [3], [4]. If \(H\) is a \(k\)-dimensional oriented submanifold of \(M\) so that the restriction of \(\xi\) on \(H\) is the volume elements on \(H\), we call \(H\) a \(\xi\)-submanifold or an integral submanifold of \(\xi\). The \(\xi\)-submanifolds have the minimizing volume in its homology class.

In the following we determine the volume-minimizing cycles of dimension 6 in Grassmann manifold \(G(2, 8)\). As is well known that \(G(2, 8)\) is a Kaehler manifold and the Euler class \(v = -\sum \omega_1^0 \land \omega_2^0\) of vector bundle
\[
E = \{(x, v) \in G(2, 8) \times R^8 \mid v \in x\} \to G(2, 8)
\]
is also the Kaehler form on \(G(2, 8)\). As is well known, \(1/k! \omega^k\) is a calibration on \(G(2, 8)\) for each \(k = 1, 2, 3\). Together with the Euler class of the vector bundle \(F\) defined below, they generate the cohomology groups \(H^*(G(2, 8))\) (see [3]).

**Theorem 7.** The submanifolds \(G(2, 5)\) and the fibres of \(\tau_1\) are the integral submanifold of the calibration \(\xi = \frac{1}{3!} \omega^3\).

**Proof.** It is easy to see that \(G(2, 5) \subset G(2, 8)\) is a \(\xi\)-submanifold. By
\[
\omega = \frac{1}{2} \sum_{l=1}^{3} [(\omega_1^{2l+1} + \omega_2^{2l+2}) \land (\omega_1^{2l+2} - \omega_2^{2l+1}) - (\omega_1^{2l+1} - \omega_2^{2l+2}) \land (\omega_1^{2l+2} + \omega_2^{2l+1})],
\]
we have \(1/2! \xi(\tilde{E}_3, \ldots, \tilde{E}_8) = 1\), where \(\tilde{E}_\alpha\) is a basis of \(T_{e_1 \wedge e_2} \tau_1^{-1}(v)\) defined in §2. These shows \(\tau_1^{-1}(v)\) is a \(\xi\)-submanifold.

Let \(J\) be a complex structure on \(R^{2m+2} \subset R^N\). Then \(\{v \wedge Jv \mid v \in S^{2m+1}\} \subset G(2, N)\) is homeomorphic to \(CP^m\). Theorem 7 can be generalized as follows:
Grassmann manifold $G(2, m + 2)$ and complex projective space $CP^m$ are two integral submanifolds of the calibration $\xi = \frac{1}{m!} \omega^m$, where $\omega$ is the Kaehler form on $G(2, N)$ ($m + 2 < N, 2m + 2 \leq N$).

Define a vector bundle on $G(2, 8)$ by
\[ p: F = \{(x, w) \in G(2, 8) \times R^8 \mid w \perp x\} \rightarrow G(2, 8), \quad p(x, w) = x. \]

Let $e_1, e_2, \ldots, e_8$ be an orthonormal frame fields on $R^8$. Then $e_3, e_4, \ldots, e_8$ can be viewed as local sections of the vector bundle $F$. The connection $\nabla$ on $F$ is defined by $\nabla e_\alpha = \sum_{\beta=3}^{8} \omega^\beta_\alpha e_\beta$ ($\alpha = 3, \ldots, 8$), and curvature forms are
\[ \Omega^\beta_\alpha = \omega^1_\alpha \wedge \omega^\beta_1 + \omega^2_\alpha \wedge \omega^\beta_2. \]
From the Euler class of $F$, we have a closed form
\[ \zeta = -\frac{1}{6!} \sum \delta^{3 \ldots 8}_{\alpha_1 \ldots \alpha_6} \Omega_{\alpha_1 \alpha_2} \Omega_{\alpha_3 \alpha_4} \Omega_{\alpha_5 \alpha_6}. \]

**Theorem 8.** The 6-form $\zeta$ is a calibration on $G(2, 8)$ and $G(1, 7) = \{\bar{e}_1 \wedge v \mid v \in S^6\}$ is a $\zeta$-submanifold.

**Proof.** It is easy to see that $-\delta^{3 \ldots 8}_{\alpha_1 \ldots \alpha_6} \Omega_{\alpha_1 \alpha_2} \Omega_{\alpha_3 \alpha_4} \Omega_{\alpha_5 \alpha_6}(X_1, \ldots, X_6) \leq 1$ holds for any orthonormal vectors $X_1, \ldots, X_6 \in TG(2, 8)$ and for any fixed $\alpha_1, \alpha_2, \ldots, \alpha_6$. Hence
\[ -\frac{1}{6!} \sum \delta^{3 \ldots 8}_{\alpha_1 \ldots \alpha_6} \Omega_{\alpha_1 \alpha_2} \Omega_{\alpha_3 \alpha_4} \Omega_{\alpha_5 \alpha_6}(X_1, \ldots, X_6) \leq 1. \]
This shows $\zeta$ is a calibration on $G(2, 8)$.

Write $\zeta$ as
\[ \zeta = \omega^3 \wedge \omega^4 \wedge \cdots \wedge \omega^8 \wedge \omega^3 \wedge \omega^4 \wedge \cdots \wedge \omega^8 + \cdots. \]
It is easy to see that the $G(1, 7) = \{\bar{e}_1 \wedge v \mid v \in S^6\}$ is a totally geodesic submanifold of $G(2, 8)$ and is a $\zeta$-submanifold.$\square$

It is easy to see that $\int_{G(2, 5)} \zeta = \int_{\tau_1^{-1}(v)} \zeta = 0$. Then $G(2, 5)$ and $\tau_1^{-1}(v)$ belong to the same homology class of $H_6(G(2, 8))$. The complex projective space $CP^3$ and $G(1, 7) \approx S^6$ are two generators of the homology group $H_6(G(2, 8))$. These gives an answer to the problem (5) in [3]. As Theorem 7, 8 can be generalized to the Grassmann manifold $G(2, 2n)$. Then the homology classes of the Grassmann manifold $G(2, N)$ can be all represented by the integral submanifolds of $G(2, N)$ for some calibrations. For the homology groups of $G(2, N)$, see the table 2.1 in [3].

Let $dV_{S^6}$ be the volume element of $S^6$. Then $\frac{1}{8} \tau_1^*(dV_{S^6})$ is a calibration on $G(2, 8)$, but there is no integral submanifold for this calibration, even locally. We can show that
\[ \frac{1}{8} \tau_1^*(dV_{S^6}) = \frac{1}{8} \prod_{l=1}^{3} (\omega_1^{2l+1} - \omega_2^{2l+2}) \wedge (\omega_1^{2l+2} + \omega_2^{2l+1}). \]
is a summand of $\frac{1}{3!}\omega^3$.

Let $dV_{S^7}$ be the volume element of $S^7$. Then $\tau_2^*(dV_{S^7})$ also determines a calibration on $G(3, 8)$ and there is also no integral submanifold for this calibration. Let $I, J, K$ be the quaternion structures on $R^8 \cong H^2$. The map

$$f: S^7 \rightarrow G(3, 8), \; v \mapsto IvJvKv$$

gives a section of the fibre bundle $\tau_2: G(3, 8) \rightarrow S^7$. We can show that $\int_{S^7} f^*(\tau_2^*(dV_{S^7})) \neq 0$. Then $f(S^7)$ is a generator of $H_7(G(3, 8))$. Similarly we can show that ASSOC is a generator of $H_8(G(3, 8))$.

From the various fibre bundles defined on the Grassmann manifold $G(3, 8)$ in §2, we can get many calibrations on $G(3, 8)$, but we can not find any integral submanifolds for them.

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