A Representation of Ring Homomorphisms on Unital Regular Commutative Banach Algebras

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Abstract

We give a complete representation of a ring homomorphism from a unital semisimple regular commutative Banach algebra into a unital semisimple commutative Banach algebra, which need not be regular. As a corollary we give a sufficient condition in order that a ring homomorphism is automatically linear or conjugate linear.

KEYWORDS: commutative Banach algebras, ring homomorphisms.
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Abstract. We give a complete representation of a ring homomorphism from a unital semisimple regular commutative Banach algebra into a unital semisimple commutative Banach algebra, which need not be regular. As a corollary we give a sufficient condition in order that a ring homomorphism is automatically linear or conjugate linear.

1. Introduction and results

Let $A$ and $B$ be two algebras. We say that a map $\rho: A \to B$ is a ring homomorphism if $\rho$ preserves both addition and multiplication. That is,

$$\rho(f + g) = \rho(f) + \rho(g),$$
$$\rho(fg) = \rho(f) \rho(g)$$

for every $f, g \in A$. Moreover if such $\rho$ preserves scalar multiplication, then we say that $\rho$ is a homomorphism.

In this paper, $C(K)$ denotes the commutative Banach algebra of all complex-valued continuous functions on a compact Hausdorff space $K$. We say that a map $\rho: C(X) \to C(Y)$ is a $*$-ring homomorphism if $\rho$ is a ring homomorphism which also preserves complex conjugate: $\rho(\overline{f}) = \overline{\rho(f)}$ for every $f \in C(X)$. Šemrl [6] made a study of $*$-ring homomorphisms on $C(X)$ into $C(Y)$ and remarked that the problem of a general description of all ring homomorphisms on $C(X)$ into $C(Y)$ is much more difficult than the problem of characterizing all $*$-ring homomorphisms. In fact, let $G$ be the set of all surjective ring homomorphisms between the complex number field $\mathbb{C}$. It is well-known that the cardinal number of $G$ is $2^c$ (cf. [1]). Here $c$ denotes the cardinal number of $\mathbb{C}$.

Let $A$ be a unital regular semisimple commutative Banach algebra and $B$ a unital semisimple commutative Banach algebra, which need not be regular. In this paper, we consider a ring homomorphism $\rho: A \to B$ and give a representation of $\rho$; hence a description of a ring homomorphism on $C(X)$ into $C(Y)$ is given. This is an answer to the Šemrl’s remark above. As a corollary, we can show [5, Theorem 1] and a unital version of [6, Theorem 5.2]. We also prove that an injective or a surjective ring

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homomorphism on $A$ to $B$ is linear or conjugate linear if the maximal ideal spaces of $A$ and $B$ are both infinite and if every constant function is mapped to a constant function.

Throughout this note, $A$ and $B$ denote a unital regular semisimple commutative Banach algebra and a unital semisimple commutative Banach algebra with the maximal ideal spaces $M_A$ and $M_B$, respectively. The units of $A$ and $B$ are denoted by the same symbol $e$. We simply write $f$ for the Gelfand transform of $f$. Before we state our main theorem, we need some terminologies.

Definition 1.1. Let $\rho: A \to B$ be a ring homomorphism. For each $y \in M_B$ we define the induced ring homomorphism $\rho_y: A \to \mathbb{C}$ and $\tilde{\rho}_y: \mathbb{C} \to \mathbb{C}$ as

$$\rho_y(f) = \rho(f)(y) \quad (f \in A),$$

$$\tilde{\rho}_y(z) = \rho(ze)(y) \quad (z \in \mathbb{C}).$$

Moreover, $q_y: A \to A/\ker \rho_y$ denotes the quotient map for every $y \in M_B$.

A decomposition of a topological space $T$ is a family $\{T_1, T_2, \ldots, T_n\}$ of finitely many subsets $T_1, T_2, \ldots, T_n \subset T$ with the following properties:

$$T = \bigcup_{j=1}^k T_j \quad \text{and} \quad T_j \cap T_k = \emptyset \text{ if } j \neq k.$$  

Note that each $T_j$ need not be clopen.

Let $A$ be a commutative algebra with unit. Recall that $\mathcal{P}$ is a prime ideal of $A$ if $\mathcal{P}$ is a proper ideal which satisfies that $fg \in \mathcal{P}$ implies $f \in \mathcal{P}$ or $g \in \mathcal{P}$. Here and after the term ideal will mean algebra ideal. In particular, every maximal ideal is a prime ideal. By Lemma 2.2, we see that the kernel $\ker \rho_y$ of the map $\rho_y: A \to \mathbb{C}$ is a prime ideal if $\ker \rho_y \neq A$. Hence, the quotient algebra $A/\ker \rho_y$ is an integral domain. Therefore, we can define the quotient field $\mathcal{F}_y$ of $A/\ker \rho_y$ if $\ker \rho_y \neq A$.

Now we are in a position to state our results.

Theorem 1.1. Let $\rho: A \to B$ be a ring homomorphism. Then there exist a decomposition $\{M_{-1}, M_0, M_1, M_m, M_p\}$ of $M_B$ and a continuous map $\Phi: M_B \setminus M_0 \to M_A$ with the following property:

For every $y \in M_m \cup M_p$ there exists a non-zero field homomorphism $\tau_y: \mathcal{F}_y \to \mathbb{C}$ such that

$$\rho(f)(y) = \begin{cases} f(\Phi(y)) & y \in M_{-1} \\ 0 & y \in M_0 \\ f(\Phi(y)) & y \in M_1 \\ \tau_y(f(\Phi(y))) & y \in M_m \\ \tau_y(q_y(f)) & y \in M_p \end{cases}$$

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for every $f \in A$.

Moreover, if $\rho$ is surjective then the map $\Phi$ is an injection defined on $M_B$ into $M_A$.

**Corollary 1.2.** Let $\rho: A \to B$ be an injective or a surjective ring homomorphism satisfying $\rho(Ce) \subset Ce$. If $M_A$ and $M_B$ are both infinite, then $\rho$ is linear or conjugate linear.

**Recall** that a subset $S$ of $C(X)$ is separating if for each $x, y \in X$ with $x \neq y$ there corresponds an $f \in S$ so that $f(x) \neq f(y)$. We say that $S$ vanishes nowhere if for every $x \in X$ there exists a function $g$ of $S$ such that $g(x) \neq 0$.

**Corollary 1.3** (cf. Molnar, [5]). Let $\rho: C(X) \to C(Y)$ be a ring homomorphism whose range contains a separating subalgebra of $C(Y)$. If the range $\rho(C(X))$ vanishes nowhere, then $\rho$ is surjective.

**Corollary 1.4** (Šemrl, [6]). Let $\rho: C(X) \to C(Y)$ be a $\ast$-ring homomorphism. Then there exist a clopen decomposition $\{Y_{-1}, Y_0, Y_1\}$ of $Y$ and a continuous map $\Phi: Y_{-1} \cup Y_1 \to X$ such that

$$\rho(f)(y) = \begin{cases} f(\Phi(y)) & y \in Y_{-1} \\ 0 & y \in Y_0 \\ f(\Phi(y)) & y \in Y_1 \end{cases}$$

for every $f \in C(X)$.

2. **Lemmas**

Let $\tau: \mathbb{C} \to \mathbb{C}$ be a ring homomorphism. We simply say that $\tau$ is a ring homomorphism on $\mathbb{C}$. For example, $\tau(z) = 0$ ($z \in \mathbb{C}$), $\tau(z) = z$ ($z \in \mathbb{C}$) and $\tau(z) = \overline{z}$ ($z \in \mathbb{C}$) are ring homomorphisms on $\mathbb{C}$; we call them trivial ring homomorphisms.

**Proposition 2.1.** Let $\tau$ be a ring homomorphism on $\mathbb{C}$. Then the following conditions are equivalent.

(i) $\tau$ is trivial.

(ii) There exist $m_0, L_0 > 0$ such that $|z| < m_0$ implies $|\tau(z)| \leq L_0$.

(iii) $\tau$ is continuous at 0.

(iv) $\tau$ is continuous at every point of $\mathbb{C}$.

(v) $\tau$ preserves complex conjugate.
Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) It is enough to consider the case where $\tau$ is non-zero. Then by a simple calculation, we see that $\tau(r) = r$ for every $r \in \mathbb{Q}$, the rational number field of real numbers. For every $\varepsilon > 0$ fix an $r_0 \in \mathbb{Q}$ with $L_0 < r_0 \varepsilon$. If $|z| < m_0/r_0$ then we have $|\tau(r_0 z)| \leq L_0$ by hypothesis. Since $\tau$ fixes every rational number, we obtain $|\tau(z)| \leq L_0/r_0 < \varepsilon$ if $|z| < m_0/r_0$. Thus $\tau$ is continuous at 0.

(iii) ⇒ (iv) Let $\{z_n\}$ be a sequence converging to $z$. Since $\tau$ is continuous at 0, we see that $\tau(z_n - z) \to 0$ as $n \to \infty$. Hence $\tau(z_n)$ converges to $\tau(z)$.

(iv) ⇒ (v) We consider the case where $\tau$ is non-zero. Then $\tau(r) = r$ for every $r \in \mathbb{Q}$. Since $\tau$ is continuous, we have that $\tau(t) = t$ for every $t \in \mathbb{R}$, the real number field. We also have that $\tau(i) = \pm i$ since $\tau(-1) = -1$. This implies that $\tau(z) = \bar{\tau(z)}$ for every $z \in \mathbb{C}$.

(v) ⇒ (i). By hypothesis, we have $\tau(\mathbb{R}) \subset \mathbb{R}$, and hence $\tau(x + h^2) - \tau(x) = \{\tau(h)\}^2 \geq 0$ for every $x, h \in \mathbb{R}$. It follows that $\tau(x) \geq \tau(y)$ for $x, y \in \mathbb{R}$ with $x \geq y$. If $\tau$ is non-zero, then $\tau$ fixes all $r \in \mathbb{Q}$. Therefore, we obtain $\tau(x) = x$ for $x \in \mathbb{R}$, so that $\tau$ is trivial.

As remarked in the previous section, there exist non-trivial ring homomorphisms on $\mathbb{C}$. By Proposition 2.1, non-trivial ring homomorphisms are discontinuous at each point of $\mathbb{C}$. Moreover a non-trivial ring homomorphism $\tau$ on $\mathbb{C}$ has the following property:

For every pair $m, L > 0$ there exists a $z \in \mathbb{C}$ such that $|z| < m$ but $|\tau(z)| > L$.

It is well-known that the kernels of non-zero complex homomorphisms on a unital commutative Banach algebra are maximal ideals. Let $\mathbb{N}$ be the space of all natural numbers and $K_0 = \{0\} \cup \{1/n; n \in \mathbb{N}\}$ with its usual topology. Šemrl showed the existence of a non-zero complex ring homomorphism $\varphi$ on $C(K_0)$ whose kernel ker $\varphi$ is not a maximal ideal of $C(K_0)$ ([6, Example 5.4]). We show that the kernel ker $\varphi$ of a non-zero complex ring homomorphism $\varphi$ on $A$ is a prime ideal that is contained in a unique maximal ideal. De Marco and Orsatti [4] gave a characterization of a commutative ring with unit of which each prime ideal containing the Jacobson radical is contained in a unique maximal ideal.

Lemma 2.2. Let $\phi: A \to \mathbb{C}$ be a non-zero ring homomorphism. Then the kernel ker $\phi$ is a prime ideal which is contained in a unique maximal ideal of $A$.

Proof. As a first step, we show that ker $\phi$ is an ideal of $A$. Since $\phi$ preserves both addition and multiplication, it is enough to show that $zf$ belongs to ker $\phi$ for every $z \in \mathbb{C}$ and $f \in$ ker $\phi$. Note that $\phi(e) = 1$ since $\phi$ is non-zero.
Therefore, we have
\[ \phi(z f) = \phi(z)\phi(e) = \phi(f)\phi(ze) = 0 \]
for every \( z \in \mathbb{C} \) and \( f \in \ker \phi \). Hence \( \ker \phi \) is an ideal of \( A \). It is now obvious that \( \ker \phi \) is a prime ideal.

Since \( \ker \phi \) is a proper ideal, there corresponds an \( x_0 \in M_A \) such that \( \ker \phi \subset \{ f \in A; f(x_0) = 0 \} \). We show that \( \{ f \in A; f(x_0) = 0 \} \) is the unique maximal ideal containing \( \ker \phi \). To this end, assume to the contrary that there exists an \( x_1 \in M_A \) such that \( x_0 \neq x_1 \) and \( \ker \phi \subset \{ f \in A; f(x_1) = 0 \} \). Let \( V_j \) be an open neighborhood of \( x_j \) for \( j = 0, 1 \) so that \( V_0 \cap V_1 = \emptyset \). Since \( A \) is regular, there corresponds an \( f_j \in A \) such that
\[ f_j(x_j) = 1 \quad \text{and} \quad f_j(A/\ker \phi, V_j) = 0 \quad (j = 0, 1). \]
Then \( f_0 f_1 = 0 \) on \( M_A \). Since \( \ker \phi \) is a prime ideal, \( f_0 \) or \( f_1 \) belongs to \( \ker \phi \). This is a contradiction since \( f_j(x_j) = 1 \) for \( j = 0, 1 \). Hence \( \ker \phi \) is contained in the unique maximal ideal \( \{ f \in A; f(x_0) = 0 \} \).

**Lemma 2.3.** Let \( \phi: A \to \mathbb{C} \) be a non-zero ring homomorphism and \( q: A \to A/\ker \phi \) the quotient map. Then \( \phi \) is of the form \( \phi = \tau \circ q \) for some non-zero field homomorphism \( \tau \) on the quotient field \( \mathcal{F} \) of \( A/\ker \phi \). If, in addition, \( \ker \phi \) is a maximal ideal of \( A \), then we may consider \( \tau \) a non-zero ring homomorphism on \( \mathbb{C} \) and \( q \in M_A \).

**Proof.** Note that the quotient field \( \mathcal{F} \) of \( A/\ker \phi \) is well-defined since \( \ker \phi \) is a prime ideal of \( A \), by Lemma 2.2. We define the map \( \tau: \mathcal{F} \to \mathbb{C} \) by
\[
(\#) \quad \tau([f]/[g]) = \frac{\rho(f)}{\rho(g)} \quad ([f]/[g] \in \mathcal{F}).
\]
Here \([f] \in A/\ker \phi \) denotes the equivalence class of \( f \in A \) with respect to \( \ker \phi \). Then \( \tau \) is a well-defined non-zero field homomorphism on \( \mathcal{F} \). If we identify \([f]\) with \([f]/[e]\), it is obvious that \( \phi \) is of the form \( \phi = \tau \circ q \).

Moreover if \( \ker \phi \) is a maximal ideal of \( A \), then the quotient algebra \( A/\ker \phi \) is isometrically isomorphic to \( \mathbb{C} \). Thus, we may identify \( A/\ker \phi \) with the quotient field \( \mathcal{F} \) of \( A/\ker \phi \). Let \( I \) be the isomorphism on \( A/\ker \phi \) onto \( \mathbb{C} \). Then \( \tau \circ I^{-1} \) is a ring homomorphism on \( \mathbb{C} \) and \( I \circ q \) a non-zero complex homomorphism on \( A \) with \( \rho = \tau \circ q = (\tau \circ I^{-1}) \circ (I \circ q) \). This completes the proof. 

**Definition 2.1.** Let \( \rho: A \to B \) be a ring homomorphism. Put \( M_0 = \{ y \in M_B; \ker \rho_y = A \} \). We define the subsets \( M_{B(m)} \) and \( M_{B(p)} \) of \( M_B \setminus M_0 \) as
\[
M_{B(m)} = \{ y \in M_B \setminus M_0; \ker \rho_y \text{ is a maximal ideal of } A \},
\]
\[
M_{B(p)} = \{ y \in M_B \setminus M_0; \ker \rho_y \text{ is not a maximal ideal of } A \}.
\]
Let $M_{-1}$, $M_1$, $M_{m,-1}$ and $M_{m,1}$ be as follows:

- $M_{-1} = \{ y \in M_B(m) ; \tilde{\rho}_y(z) = 0 \}$,
- $M_1 = \{ y \in M_B(m) ; \tilde{\rho}_y(z) = z \}$,
- $M_{m,-1} = \{ y \in M_B(m) ; \tilde{\rho}_y$ is non-trivial and $\tilde{\rho}_y(i) = -i \}$,
- $M_{m,1} = \{ y \in M_B(m) ; \tilde{\rho}_y$ is non-trivial and $\tilde{\rho}_y(i) = i \}$.

The subsets $M_{p,-1}$ and $M_{p,1}$ of $M_B(p)$ are defined by

- $M_{p,-1} = \{ y \in M_B(p) ; \tilde{\rho}_y(i) = -i \}$,
- $M_{p,1} = \{ y \in M_B(p) ; \tilde{\rho}_y(i) = i \}$.

Then we write $M_{d,j} = M_{m,j} \cup M_{p,j}$ ($j = -1, 1$) and $M_d = M_{d,-1} \cup M_{d,1}$.

Note that $\tilde{\rho}_y$ is a non-trivial ring homomorphism on $\mathbb{C}$ for every $y \in M_d$. For if $\tilde{\rho}_y$ is trivial then

$$
\rho_y(zf) = \tilde{\rho}_y(z)\rho_y(f) \quad (z \in \mathbb{C}, f \in A)
$$

implies that $\ker \rho_y$ is maximal for every $y \in M_B \setminus M_0$. By definition, the subsets $M_{-1}$, $M_0$, $M_1$ and $M_d$ of $M_B$ are mutually disjoint and $M_B = M_{-1} \cup M_0 \cup M_1 \cup M_d$. Hence, $\{M_{-1}, M_0, M_1, M_d\}$ above is a decomposition of $M_B$. We call $\{M_{-1}, M_0, M_1, M_d\}$ the decomposition of $M_B$ with respect to $\rho$.

Until the end of this section, $\rho : A \to B$ denotes a ring homomorphism and $\{M_{-1}, M_0, M_1, M_d\}$ the decomposition of $M_B$ with respect to $\rho$.

**Lemma 2.4.** The sets $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen in $M_B$. Also $M_{-1}$ and $M_1$ are both closed in $M_B$.

**Proof.** By definition, it is easy to see that

- $M_0 = \{ y \in M_B ; \tilde{\rho}_y(i) = 0 \}$,
- $M_{-1} \cup M_{d,-1} = \{ y \in M_B ; \tilde{\rho}_y(i) = -i \}$,
- $M_1 \cup M_{d,1} = \{ y \in M_B ; \tilde{\rho}_y(i) = i \}$.

Therefore, $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen since the function $\rho(ie)$ is continuous on $M_B$.

Next, we show that $M_1$ is closed in $M_B$. For every $y \in M_{d,1}$ we can find a $z_0 \in \mathbb{C}$ such that $\tilde{\rho}_y(z_0) \neq z_0$ since $\tilde{\rho}_y$ is non-trivial. Put

$$
V = \{ w \in M_B ; |\rho(z_0e)(w) - \rho(z_0e)(y)| < |z_0 - \tilde{\rho}_y(z_0)|/2 \}.
$$

Then $V$ is an open neighborhood of $y$ with $V \cap M_1 = \emptyset$. Since $M_1 \cup M_{d,1}$ is clopen, this implies that $M_1$ is closed. In a way similar to the above, we see that $M_{-1}$ is closed and the proof is omitted. \qed
**Definition 2.2.** By Lemma 2.2, for every \( y \in M_B \setminus M_0 \) there exists a unique \( x \in M_A \) such that \( \ker \rho_y \subset \{ f \in A; f(x) = 0 \} \). We denote the correspondence defined on \( M_B \setminus M_0 \) into \( M_A \) as \( \Phi \); That is, \( \ker \rho_y \) is contained in the unique maximal ideal \( \{ f \in A; f(\Phi(y)) = 0 \} \) for every \( y \in M_B \setminus M_0 \). We call \( \Phi \) the representing map for \( \rho \).

**Lemma 2.5.** Let \( r \in \mathbb{Q} \), \( G \) open in \( M_A \) and \( \Phi \) the representing map for \( \rho \). Suppose that \( h \in A \) satisfies \( h(G) = r \) then \( \rho_y(h) = r \) for every \( y \in \Phi^{-1}(G) \).

**Proof.** Put \( h_r = h - re \in A \) and fix \( y \in \Phi^{-1}(G) \). Since \( A \) is regular, there exists a function \( g \in A \) such that \( g(\Phi(y)) = 1 \) and \( g(M_A \setminus G) = 0 \). Then \( gh_r = 0 \) on \( M_A \). Since \( \ker \rho_y \) is a prime ideal, \( g \) or \( h_r \) belongs to \( \ker \rho_y \). On the other hand, \( g \) does not belong to \( \{ f \in A; f(\Phi(y)) = 0 \} \) since \( g(\Phi(y)) = 1 \). So we conclude that \( h_r \in \ker \rho_y \). Therefore we have \( \rho_y(h) = r \) for every \( y \in \Phi^{-1}(G) \).

**Lemma 2.6.** Let \( \Phi \) be the representing map for \( \rho \). Then the range \( \Phi(M_d) \) is at most finite.

**Proof.** Assume to the contrary that \( \Phi(M_d) \) has a countable subset \( \{ x_n \}_{n=1}^\infty \) such that \( x_j \neq x_k \) if \( j \neq k \). Without loss of generality, we may assume that each \( x_j \) is an isolated point of \( \{ x_n \}_{n=1}^\infty \). By definition, for every \( n \in \mathbb{N} \) there exists a \( y_n \in M_d \) such that \( x_n = \Phi(y_n) \). By induction, we can find an open neighborhood \( U_j \) of \( x_j \) with

\[
(\overline{U}_j \setminus \{ x_j \}) \cap \{ x_n \}_{n=1}^\infty = \emptyset \quad \text{and} \quad \overline{U}_{j+1} \subset M_A \setminus \bigcup_{k=1}^j \overline{U}_k
\]

for every \( j \in \mathbb{N} \). Here \( \overline{U}_j \) denotes the closure of \( U_j \) in \( M_A \). Let \( V_j \) be an open neighborhood of \( x_j \) so that \( \overline{V}_j \subset U_j \). Since \( A \) is regular, \( A \) is normal (cf. [2, Theorem 6.3 of Chapter I]). That is, there exists a \( g_j \in A \) such that \( g_j(\overline{V}_j) = 1 \) and \( g_j(M_A \setminus U_j) = 0 \). Since \( \tilde{\rho}_{y_j} \) is non-trivial, there corresponds a \( z_j \in \mathbb{C} \) so that

\[
|z_j| < (2^j ||g_j||)^{-1} \quad \text{and} \quad |\tilde{\rho}_{y_j}(z_j)| > 2^j,
\]

by Proposition 2.1. Here \( || \cdot || \) denotes the Banach norm on \( A \). Put \( f_j = z_jg_j \in A \). Then \( \rho_y(f_j) = \tilde{\rho}_{y_j}(z_j)\rho_y(g_j) \) for every \( y \in M_B \). Therefore, by Lemma 2.5 we see that \( \rho_{y_j}(f_j) = \tilde{\rho}_{y_j}(z_j) \). Since \( ||f_j|| < 2^{-j} \), the series \( \sum_{n=1}^\infty f_n \) converges in \( A \), say \( f_0 \). Note that \( f_j = 0 \) on \( V_k \) if \( k \neq j \). Thus we see that \( f_0 = f_j \) on \( V_j \) for every \( j \in \mathbb{N} \). By Lemma 2.5, we obtain \( \rho_{y_j}(f_0 - f_j) = 0 \). Therefore,

\[
|\rho_{y_j}(f_0)| = |\rho_{y_j}(f_j)| = |\tilde{\rho}_{y_j}(z_j)| > 2^j \quad (j \in \mathbb{N}).
\]

This is a contradiction since \( \rho(f_0) \) is bounded on \( M_B \). Hence we have proved that the range \( \Phi(M_d) \) is at most finite. \( \square \)
3. A Proof of Main Result

Proof of Theorem 1.1. Let \( \{M_{-1}, M_0, M_1, M_d\} \) and \( \Phi \) be the decomposition of \( M_B \) with respect to \( \rho \) and the representing map for \( \rho \), respectively. For every \( y \in M_B \setminus M_0 \), let \( q_y : A \rightarrow A/\ker \rho_y \) denote the quotient map. Recall that \( M_{B(m)} \) is the set of all \( y \in M_B \) so that \( \ker \rho_y \) is a maximal ideal of \( A \). By Lemma 2.3, we can find a field homomorphism \( \tau_y \) on the quotient field \( F_y \) of the integral domain \( A/\ker \rho_y \) into \( \mathbb{C} \) such that \( \rho_y = \tau_y \circ q_y \). If, in addition, \( y \in M_{B(m)} \), then we may consider that \( \tau_y \) is a ring homomorphism on \( \mathbb{C} \) and \( q_y \in M_A \). In this case, we therefore have \( \ker q_y = \ker \rho_y = \ker \Phi(y) \). Hence, we see that \( q_y = \Phi(y) \) for every \( y \in M_{B(m)} \). By the formula (\( \ast \)), we also have \( \tau_y = \hat{\rho}_y \) for every \( y \in M_{B(m)} \). That is, \( \tau_y(z) = \bar{z} \) if \( y \in M_{-1} \), \( \tau_y(z) = z \) if \( y \in M_1 \) and \( \tau_y \) is non-trivial if \( y \in M_{m,-1} \cup M_{m,1} \). Therefore, we have

\[
\rho(f)(y) = \begin{cases} 
0 & y \in M_0 \\
\tau_y(f(\Phi(y))) & y \in M_{B(m)} \\
\tau_y(q_y(f)) & y \in M_{B(p)} \\
\frac{f(\Phi(y))}{\tau_y(f(\Phi(y)))} & y \in M_{-1} \\
0 & y \in M_0 \\
f(\Phi(y)) & y \in M_1 \\
\tau_y(f(\Phi(y))) & y \in M_{m,-1} \cup M_{m,1} \\
\tau_y(q_y(f)) & y \in M_{p,-1} \cup M_{p,1}
\end{cases}
\]

for every \( f \in A \).

By Lemma 2.6, we may put \( \Phi(M_d) = \{x_1, x_2, \ldots, x_m\} \). Then we see that the set \( M_d(x_j) = \{y \in M_d \mid \Phi(y) = x_j\} \) is open in \( M_B \) for \( j = 1, 2, \ldots, m \). Indeed, assume to the contrary that \( M_d(x_j) \) is not open. Then there exist a \( y_j \in M_d(x_j) \) and a net \( \{y_\alpha\} \) in \( M_B \setminus M_d(x_j) \) such that \( y_\alpha \) converges to \( y_j \).

Since \( M_{-1} \cup M_0 \cup M_1 \) is closed in \( M_B \) by Lemma 2.4, we see that \( M_d \) is an open subset of \( M_B \). Therefore, without loss of generality we may assume \( \{y_\alpha\} \subset M_d \setminus M_d(x_j) \). Fix open neighborhoods \( O_1, O_2 \) of \( x_j \) with \( \overline{O}_1 \subset O_2 \) and \( \overline{O}_2 \cap \Phi(M_d) = \{x_j\} \). Here, \( \overline{\tau} \) denotes the closure in \( M_A \). Since \( A \) is regular, we can find a function \( h_j \in A \) so that \( h_j(\overline{O}_1) = 1 \) and \( h_j(M_A \setminus O_2) = 0 \).

By Lemma 2.5, we have that \( \rho_{y_\alpha}(h_j) = 1 \) and \( \rho_{y_\alpha}(h_j) = 0 \) for every \( \alpha \). This is a contradiction since \( \rho(h_j) \) is continuous on \( M_B \). Therefore, the set \( M_d(x_j) = \{y \in M_d \mid \Phi(y) = x_j\} \) is open in \( M_B \) for \( j = 1, 2, \ldots, m \).

Finally we show that the map \( \Phi \) on \( M_B \setminus M_0 \) into \( M_A \) is continuous. Indeed, we see that \( \Phi \) is continuous at each \( y_0 \in M_d \) since \( M_d(\Phi(y_0)) = \{y \in M_d \mid \Phi(y) = \Phi(y_0)\} \) is open as proved above. We show that \( \Phi \) is continuous on \( M_{-1} \cup M_1 \). Let \( y_1 \) be a point of \( M_1 \) and \( \{y_\beta\} \in \mathcal{F} \) an arbitrary net in \( M_B \setminus M_0 \) converging to \( y_1 \). Since \( M_0 \cup M_{-1} \) is closed in \( M_B \), we see...
that $M \cup M_d$ is an open subset of $M_B$. Hence, without loss of generality we may assume \( \{y_\beta\}_{\beta \in \Gamma} \subset M \cup M_d \). We assert that there exists a $\beta_0 \in \Gamma$ such that $y_\beta \in M \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\}$ for every $\beta \in \Gamma$ with $\beta \geq \beta_0$.

In fact, let $W_1$ be an open neighborhood of $\Phi(y_1)$ and $W_2$ an open subset containing $\Phi(M_d) \setminus \{\Phi(y_1)\}$ such that $W_1 \cap W_2 = \emptyset$. Then we can find a $g_0 \in A$ such that $g_0(W_1) = 1$ and $g_0(W_2) = 0$. By Lemma 2.5, we see that $\rho(y_1)g_0 = 1$ and $\rho_g(y_1) = 0$ for every $y \in \Phi^{-1}(W_2)$. By the continuity of $\rho(y_1)$, there exists a $\beta_0 \in \Gamma$ such that $\beta \geq \beta_0$ implies $|\rho(g_0)(y_\beta) - 1| < 1/2$. That is, $\Phi(y_\beta) \not\in \Phi(M_d) \setminus \{\Phi(y_1)\}$ if $\beta \geq \beta_0$. Therefore, we see that $y_\beta \in M \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\}$ for every $\beta \in \Gamma$ with $\beta \geq \beta_0$. Hence, if $\beta \geq \beta_0$ then we have

$$f(\Phi(y_\beta)) = \begin{cases} \rho(f)(y_\beta) & y_\beta \in M \\ f(\Phi(y_1)) & \Phi(y_\beta) = \Phi(y_1) \end{cases}$$

for every $f \in A$. Consequently, $\beta \geq \beta_0$ implies that

$$|f(\Phi(y_\beta)) - f(\Phi(y_1))| \leq |\rho(f)(y_\beta) - \rho(f)(y_1)|$$

for every $f \in A$. Thus $\Phi(y_\beta)$ converges to $\Phi(y_1)$. This implies that $\Phi$ is continuous on $M_1$. In a way similar to the above, we can show that $\Phi$ is continuous on $M_{-1}$ and the proof is omitted. Thus, we have proved that the map $\Phi$ is continuous on $M_B \setminus M_0$.

Suppose that $\rho$ is surjective. Then $M_0$ is an empty set. Hence $\Phi$ is the map defined on $M_B$ into $A$. We show that $\ker \rho_y = \{f \in A; f(\Phi(y)) = 0\}$. Recall that $\ker \rho_y \subset \{f \in A; f(\Phi(y)) = 0\}$. So it is enough to show that $\rho_y(f) \neq 0$ implies $f(\Phi(y)) \neq 0$. Let $a \in A$ satisfy $\rho_y(a) \neq 0$. Since $\rho_y(A) = \mathbb{C}$, there corresponds a $b \in A$ such that $\rho_y(a)\rho_y(b) = 1$. Therefore, $ab - e$ belongs to $\ker \rho_y$. We conclude that $a(\Phi(y)) \neq 0$ since $(ab - e)(\Phi(y)) = 0$. Thus, we have proved that $\rho_y = \{f \in A; f(\Phi(y)) = 0\}$. Hence $M_B = M_{-1} \cup M_1 \cup M_{m,-1} \cup M_{m,1}$.

Let $w_1, w_2 \in M_B$ satisfy $w_1 \neq w_2$. Since $\rho$ is surjective, there exists an $a_0 \in A$ such that $\rho(a_0)(w_1) = 1$ and $\rho(a_0)(w_2) = 0$. By the formula for $\rho$, it is easy to see that

$$a_0(\Phi(w_1)) = 1 \quad a_0(\Phi(w_2)) = 0.$$ 

Therefore, we have $\Phi(w_1) \neq \Phi(w_2)$. This implies that $\Phi$ is injective. \( \square \)

**Proof of Corollary 1.2.** Let \( \{M_{-1}, M_0, M_1, M_{d,-1}, M_{d,1}\} \) be the decomposition of $M_B$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since $\rho(\mathbb{C}e) \subset \mathbb{C}e$, we have $M_B = M_{-1} \cup M_{d,-1}$ or $M_B = M_0$ or $M_B = M_1 \cup M_{d,1}$. It is enough to consider the case where $M_B = M_{-1} \cup M_{d,-1}$ or $M_B = M_1 \cup M_{d,1}$.

Suppose that $M_B = M_1 \cup M_{d,1}$. First, we show that $M_1 \neq \emptyset$. Suppose not. Then $M_B = M_{d,1}$. If $\rho$ is surjective, the map $\Phi$ is injective by Theorem 1.1. Since $\Phi(M_{d,1})$ is finite by Lemma 2.6, so is $M_{d,1} = M_B$. This is a
contradiction. Therefore, $M_1 \neq \emptyset$ if $\rho$ is surjective. Consider the case where $\rho$ is injective. Since $M_A$ is infinite, there exists an $x_0 \in M_A \setminus \Phi(M_{d,1})$. We can find an open subset $V$ of $M_A$ so that $\Phi(M_{d,1}) \subset V$ and $x_0 \notin V$. Since $A$ is regular, there corresponds an $f_0 \in A$ such that $f_0(x_0) = 1$ and $f_0(\bar{V}) = 0$. By Lemma 2.5 we see that $\rho_y(f_0) = 0$ for every $y \in M_{d,1} = M_B$. Since $f_0$ is not identically zero, this contradicts that $\rho$ is injective. Consequently, we have that $M_1 \neq \emptyset$.

Now we show that $M_B = M_1$. Suppose that there exists a $y_1 \in M_{d,1}$. Since $\tilde{\rho}_{y_1}$ is non-trivial, we can find a $z_1 \in C$ such that $\tilde{\rho}_{y_1}(z_1) \neq z_1$. Note that $\tilde{\rho}_{y}(z_1) = z_1$ for every $y \in M_1$. This is a contradiction since $\rho(Ce) \subset Ce$. Therefore, we have proved that $M_B = M_1$ if $M_B = M_1 \cup M_{d,1}$. In a way similar to the above, we see that $M_B = M_{-1}$ if $M_B = M_{-1} \cup M_{d,-1}$. Hence, $\rho$ is linear or conjugate linear.

\begin{proof}[Proof of Corollary 1.3] Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since the range $\rho(C(X))$ vanishes nowhere, we see that $Y_0$ is an empty set. Since $\rho(C(X))$ contains a separating subalgebra, in a way similar to the proof of Theorem 1.1, we can prove that $\ker \rho_y$ is a maximal ideal for every $y \in Y$ and that $\Phi : Y \to X$ is injective. Hence, $Y$ is homeomorphic to the range $\Phi(Y)$. Let $\varphi : \Phi(Y) \to Y$ be the homeomorphism defined by

$$\varphi(x) = \Phi^{-1}(x) \quad (x \in \Phi(Y)).$$

Note that

$$\rho(f)(y) = \begin{cases} f(\Phi(y)) & y \in Y_{-1} \\ f(\Phi(y)) & y \in Y_1 \\ \tau_y f(\Phi(y)) & y \in Y_d \end{cases}$$

for every $f \in C(X)$. Here $\tau_y$ denotes a non-trivial ring homomorphism on $C$. We define the continuous function $h : \Phi(Y) \to \mathbb{C}$ by

$$h(x) = \begin{cases} \tilde{\varphi}(\varphi(x)) & x \in \Phi(Y_{-1}) \\ \varphi(x) & x \in \Phi(Y_1) \\ \tau_{\varphi(x)^{-1}}(\varphi(x)) & x \in \Phi(Y_d) \end{cases}$$

for each $g \in C(Y)$. Since $\Phi(Y_{-1})$, $\Phi(Y_1)$ and $\Phi(Y_d)$ are disjoint closed subsets of the compact Hausdorff space $X$, there exists an $\tilde{h}$ of $C(X)$ such that $\tilde{h}|_{\Phi(Y)} = h$. Then it is easy to see that $\rho(\tilde{h}) = g$. Hence $\rho$ is surjective.
\end{proof}

\begin{proof}[Proof of Corollary 1.4] Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since $\rho$ preserves complex conjugate, by Proposition 2.1 we have that $\tilde{\rho}_y$ is trivial for every $y \in Y$.
\end{proof}
Therefore, $Y_d$ is an empty set. By Lemma 2.4, we see that $Y_{-1}$, $Y_0$ and $Y_1$ are all clopen. This completes the proof. □

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