Some Results on \((\sigma,\tau)\)-Lie Ideals

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Abstract

In this note we give some basic results on one sided$(\sigma, \tau)$-Lie ideals of prime rings with characteristic not 2.

KEYWORDS: Prime ring, $(\sigma, \tau)$-Lie ideal, $(\sigma, \tau)$-derivation
SOME RESULTS ON (σ, τ)-LIE IDEALS

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ABSTRACT. In this note we give some basic results on one sided (σ, τ)–Lie ideals of prime rings with characteristic not 2.

1. Introduction

Let $R$ be a ring and $σ, τ$ be two mappings from $R$ into itself. We write $[x, y] = xy - yx$, and $[x, y]_{σ, τ} = xσ(y) - τ(y)x$ for $x, y ∈ R$. For subsets $A, B ⊂ R$, let $[A, B]$ be the additive subgroup generated by all $[a, b]$, and $[A, B]_{σ, τ}$ be the additive subgroup generated by all $[a, b]_{σ, τ}$ for $a ∈ A$ and $b ∈ B$. We recall that a Lie ideal $L$ is an additive subgroup of $R$ such that $[R, L] ⊂ L$. We first introduce the generalized Lie ideal in [3] as follows. Let $U$ be an additive subgroup of $R$. (i) $U$ is called a $(σ, τ)$–right Lie ideal of $R$ if $[U, R]_{σ, τ} ⊂ U$. (ii) $U$ is called a $(σ, τ)$–left Lie ideal if $[R, U]_{σ, τ} ⊂ U$. (iii) $U$ is called a $(σ, τ)$–Lie ideal if $U$ is both a $(σ, τ)$–right and a $(σ, τ)$–left Lie ideal. An additive mapping $d : R → R$ is called a $(σ, τ)$–derivation if $d(xy) = d(x)σ(y) + τ(x)d(y)$ for all $x, y ∈ R$. We write $C_{σ, τ} = \{c ∈ R | σ(c(r) = τ(r)c for r ∈ R}\$, and will make extensive use of the following basic commutator identities:

$[xy, z]_{σ, τ} = x[y, z]_{σ, τ} + [x, τ(z)]y = x[y, σ(z)] + [x, z]_{σ, τ}y$

$[x, yz]_{σ, τ} = τ(y)[x, z]_{σ, τ} + [x, y]_{σ, τ}σ(z)$

Throughout the present paper, $R$ will represent a prime ring (of char $R$ ≠ 2, exclude Lemmas 1 and 2) and $σ, τ, α, β, λ, μ$ will be automorphisms of $R$. In this note, we give the following properties on prime rings and some results on one sided $(σ, τ)$–Lie ideals. Let $I$ be a nonzero ideal of $R$. (1) If $[[I, a]_{σ, τ}, b]_{α, β} = 0$ for $a, b ∈ R$, then $[τ(a), β(b)] = 0$. (2) If $[[a, I]_{σ, τ}, b]_{α, β} = 0$ for $a, b ∈ R$, then $b ∈ Z$ or $[a, τ^{-1}(β(b))]_{σ, τ} = 0$. (3) If $[b, [a, R]_{σ, τ}]_{α, β} = 0$ for $a, b ∈ R$, then $b ∈ C_{α, β}$, $a ∈ C_{σ, τ}$ or $a + τσ^{-1}(a) ∈ C_{σ, τ}$. On the other hand, in [4] Park and Jung proved that if $d : R → R$ is a nonzero $(σ, τ)$–derivation and $a ∈ R$ such that $d[R, a]_{σ, τ} = 0$, then $σ(a) + τ(a) ∈ Z$. We prove that if $d : R → R$ is a nonzero $(σ, τ)$–derivation and $a ∈ R$ such that $d[a, R]_{α, β} = 0$, then $a ∈ C_{α, β}$ or $a + βα^{-1}(a) ∈ C_{α, β}$.

2. Results

The following Lemmas 1 and 2 are generalizations of [1, Lemma 1.5].

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Lemma 1. Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $[[I, a]_{\sigma, \tau}, b]_{\alpha, \beta} = 0$, then $[\tau(a), \beta(b)] = 0$.

Proof. Let $[[I, a]_{\sigma, \tau}, b]_{\alpha, \beta} = 0$. Then we have $0 = [[\tau(a)y, a]_{\sigma, \tau}, b]_{\alpha, \beta} = [\tau(a)y, a]_{\sigma, \tau} + [\tau(a), \tau(y)]y, b]_{\alpha, \beta} = \tau(a)[y, a]_{\sigma, \tau}, b]_{\alpha, \beta} + [\tau(a), \beta(b)]y, a]_{\sigma, \tau}$ for all $y \in I$. This gives that

\[(2.1) [\tau(a), \beta(b)]y, a]_{\sigma, \tau} = 0 \text{ for all } y \in I.\]

Replacing $yr, r \in R$ by $y$ in (2.1), we get $0 = [\tau(a), \beta(b)]y[r, \sigma(a)] + [\tau(a), \beta(b)]y, a]_{\sigma, \tau}r$ and so

\[(2.2) [\tau(a), \beta(b)]y[r, \sigma(a)] = 0 \text{ for all } y \in I, r \in R.\]

Since $R$ is prime, we get

\[(2.3) [\tau(a), \beta(b)] = 0 \text{ or } a \in Z.\]

Thus, $[\tau(a), \beta(b)] = 0$ is obtained for two cases in (2.3) $\square$

Corollary 1. (1) If $I$ is a nonzero ideal of $R$ and $a \in R$ such that $[[I, a]_{\alpha, \beta} \subset C_{\lambda, \mu}$, then $a \in Z$.

(2) Let $U$ be a nonzero $(\sigma, \tau)$-right(left) Lie ideal of $R$ and $I$ a nonzero ideal of $R$. If $[[I, I]_{\sigma, \tau}, U]_{\alpha, \beta} = 0$ then $U \subset Z$.

(3) If $a \in R$ such that $[[I, I]_{\sigma, \tau}, a]_{\alpha, \beta} = 0$ then $a \in Z$.

Proof. (1) $[[I, a]_{\alpha, \beta} \subset C_{\lambda, \mu}$ implies that $[[I, a]_{\alpha, \beta}, R]_{\lambda, \mu} = 0$. By Lemma 1 we obtain that $[\beta(a), \mu(R)] = 0$. Since $\mu$ is onto, we have $\beta(a) \in Z$ and so $a \in Z$.

(2) By Lemma 1 we have $[\tau(I), \beta(U)] = 0$ and so $U \subset Z$.

(3) $[[I, I]_{\sigma, \tau}, a]_{\alpha, \beta} = 0$ implies that $[\tau(I), \beta(a)] = 0$ by Lemma 1 and so $a \in Z$. $\square$

Lemma 2. Let $I$ be a nonzero ideal of $R$. If $a, b \in R$ and $[[a, I]_{\sigma, \tau}, b]_{\alpha, \beta} = 0$, then $b \in Z$ or $[a, \tau^{-1}\beta(b)]_{\sigma, \tau} = 0$.

Proof. For any $x, y \in I$ we have

\[0 = [[a, xy]_{\sigma, \tau}, b]_{\alpha, \beta} \]
\[= [\tau(x)[a, y]_{\sigma, \tau} + [a, x]_{\sigma, \tau}\sigma(y)], b]_{\alpha, \beta} \]
\[= \tau(x)[[a, y]_{\sigma, \tau}, b]_{\alpha, \beta} + [\tau(x), \beta(b)]\sigma(y) + [a, x]_{\sigma, \tau}[\sigma(y), \alpha(b)] + [[a, x]_{\sigma, \tau}, b]_{\alpha, \beta}\sigma(y) \]

and so

\[(2.4) [\tau(x), \beta(b)]\sigma(y) + [a, x]_{\sigma, \tau}[\sigma(y), \alpha(b)] = 0 \text{ for all } x, y \in I.\]
Replacing $x$ by $rx$, $r \in R$ in (2.4) we get
\[
0 = \[\tau(rx), \beta(b)\]a, y\sigma, \tau + [a, rx]_{\sigma, \tau}[\sigma(y), \alpha(b)]
\]
\[
= \tau(r)\tau(x), \beta(b)]a, y\sigma, \tau + \tau(r)[\beta(b)]\tau(x)[a, y\sigma, \tau + \tau(r)[a, x]_{\sigma, \tau}[\sigma(y), \alpha(b)]
\]
\[
+ [a, r]_{\sigma, \tau}[\sigma(x)[\sigma(y), \alpha(b)].
\]
That is
\[
(2.5)\]
\[
[\tau(r), \beta(b)]\tau(x)[a, y]_{\sigma, \tau} + [a, r]_{\sigma, \tau}[\sigma(x)[\sigma(y), \alpha(b)] = 0 \text{ for all } x, y \in I, r \in R.
\]
If we take $\tau^{-1}\beta(b)$ instead of $r$ in (2.5) then we have
\[
(2.6)\]
\[
a, \tau^{-1}\beta(b)]_{\sigma, \tau}[\sigma(I), \alpha(b)] = 0.
\]
Since $\sigma(I) \neq 0$ an ideal of $R$ and $R$ is prime we get
\[
(2.7)\]
\[
[a, \tau^{-1}\beta(b)]_{\sigma, \tau} = 0 \text{ or } [\sigma(I), \alpha(b)] = 0.
\]
Since $R$ is prime, $[\sigma(I), \alpha(b)] = 0$ implies that $b \in Z$. Thus $[a, \tau^{-1}\beta(b)]_{\sigma, \tau} = 0$ or $b \in Z$ is obtained. \hfill \Box

**Lemma 3.** Let $U$ be a nonzero $(\sigma, \tau)$—right Lie ideal of $R$ and $a \in R$. If $[U, a]_{\alpha, \beta} = 0$, then $a \in Z$ or $U \subset C_{\sigma, \tau}$.

**Proof.** Since $[[U, R]_{\sigma, \tau}, a]_{\alpha, \beta} \subset [U, a]_{\alpha, \beta} = 0$ then we have
\[
a \in Z \text{ or } [U, \tau^{-1}\beta(a)]_{\sigma, \tau} = 0
\]
by Lemma 2. If $[U, \tau^{-1}\beta(a)]_{\sigma, \tau} = 0$ then $a \in Z$ or $U \subset C_{\sigma, \tau}$ by [6, Lemma 2]. \hfill \Box

**Theorem 1.** Let $U$ be a nonzero $(\sigma, \tau)$—right Lie ideal of $R$ and $I \neq 0$ an ideal of $R$.

1. If $a \in R$ and $[[U, I]_{\alpha, \beta}, a]_{\lambda, \mu} = 0$, then $a \in Z$ or $U \subset C_{\sigma, \tau}$.
2. If $[U, I]_{\alpha, \beta} \subset C_{\lambda, \mu}$, then $U \subset C_{\sigma, \tau}$ or $R$ is commutative.

**Proof.** (1) $[[U, I]_{\alpha, \beta}, a]_{\lambda, \mu} = 0$ implies that $a \in Z$ or $[U, \beta^{-1}\mu(a)]_{\alpha, \beta} = 0$, by Lemma 2. If $[U, \beta^{-1}\mu(a)]_{\alpha, \beta} = 0$ then $a \in Z$ or $U \subset C_{\sigma, \tau}$ by Lemma 3.

(2) Let $[U, I]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then we have $[[U, I]_{\alpha, \beta}, R]_{\lambda, \mu} = 0$. If we use (1) we get $R \subset Z$ or $U \subset C_{\sigma, \tau}$ and so $U \subset C_{\sigma, \tau}$ or $R$ is commutative. \hfill \Box

**Theorem 2.** Let $d$ be a nonzero $(\sigma, \tau)$—derivation on $R$ and $a \in R$. If $d[a, R]_{\alpha, \beta} = 0$, then $a \in C_{\alpha, \beta}$ or $a + \beta^{-1}(a) \in C_{\alpha, \beta}$.

**Proof.** For any $x, y \in R$ we have
\[
0 = d[a, xy]_{\alpha, \beta} = d(\beta(x)[a, y]_{\alpha, \beta} + [a, x]_{\alpha, \beta}\alpha(y))
\]
\[
= d\beta(x)[\sigma(a, y)_{\alpha, \beta} + \tau[a, x]_{\alpha, \beta}d\alpha(y)
\]
Replacing $x$ by $\beta^{-1}[a, z]_{\alpha, \beta}$ in the last relation we get
\[ [a, \beta^{-1}[a, z]_{\alpha, \beta}]_{\alpha, \beta}d\alpha(y) = 0 \text{ for all } y, z \in R \]
and so
\[ (2.8) \quad [a, \beta^{-1}[a, z]_{\alpha, \beta}]_{\alpha, \beta} = 0 \text{ for all } z \in R \]
by [5, Lemma 3]. Taking $zy$ for $z$ in (2.8) we get
\[ 0 = [a, \beta^{-1}[a, zy]_{\alpha, \beta}]_{\alpha, \beta} = [a, \beta^{-1}(\beta(z)[a, y]_{\alpha, \beta} + [a, z]_{\alpha, \beta} \alpha(y))]_{\alpha, \beta} \]
\[ = [a, z\beta^{-1}[a, y]_{\alpha, \beta} + \beta^{-1}[a, z]_{\alpha, \beta} \beta^{-1}\alpha(y)]_{\alpha, \beta} \]
\[ = [a, z]_{\alpha, \beta} \beta^{-1}[a, y]_{\alpha, \beta} + [a, z]_{\alpha, \beta}[a, \beta^{-1}\alpha(y)]_{\alpha, \beta} \]
which leads to
\[ (2.9) \quad [a, z]_{\alpha, \beta}(\alpha\beta^{-1}[a, y]_{\alpha, \beta} + [a, \beta^{-1}\alpha(y)]_{\alpha, \beta}) = 0 \text{ for all } y, z \in R. \]
Replacing $z$ by $zt$ in (2.9), we get
\[ (2.10) \quad [a, z]_{\alpha, \beta} = 0, \forall z \in R \text{ or } \alpha\beta^{-1}[a, y]_{\alpha, \beta} + [a, \beta^{-1}\alpha(y)]_{\alpha, \beta} = 0 \text{ for all } y \in R. \]
Hence $a \in C_{\alpha, \beta}$ or $0 = \alpha\beta^{-1}[a, y]_{\alpha, \beta} + \alpha\beta^{-1}\alpha(y) - \alpha(y)a$ for all $y \in R$. If we apply $\alpha^{-1}$ and $\beta$ to the last relation we have $\alpha\alpha(y) - \beta(y)a + \beta\alpha^{-1}(a)\alpha(y) - \beta(y)\beta\alpha^{-1}(a) = 0$ for all $y \in R$. This implies that $(a + \beta\alpha^{-1}(a))\alpha(y) - \beta(y)(a + \beta\alpha^{-1}(a)) = 0$ and so $a + \beta\alpha^{-1}(a) \in C_{\alpha, \beta}$ for all $y \in R$. Thus we obtain $a \in C_{\alpha, \beta}$ or $a + \beta\alpha^{-1}(a) \in C_{\alpha, \beta}$ by (2.10).

**Corollary 2.** If $[b, [a, R]_{\sigma, \tau}]_{\alpha, \beta} = 0$, then $a \in C_{\sigma, \tau}$ or $b \in C_{\alpha, \beta}$ or $a + \tau\sigma^{-1}(a) \in C_{\sigma, \tau}$.

**Proof.** $d(x) = [b, x]_{\alpha, \beta}$ is a $(\alpha, \beta)$-derivation on $R$. Furthermore $d([a, R]_{\sigma, \tau}) = 0$. This implies that $a \in C_{\sigma, \tau}$, $b \in C_{\alpha, \beta}$ or $a + \tau\sigma^{-1}(a) \in C_{\sigma, \tau}$ by Theorem 2.

**Theorem 3.** Let $U$ be a nonzero $(\sigma, \tau)$-right Lie ideal of $R$ and $d : R \rightarrow R$ a nonzero $(\lambda, \mu)$-derivation.

1. If $d(U) = 0$, then $v + \tau\sigma^{-1}(v) \in C_{\sigma, \tau}$ for all $v \in U$.
2. If $d(U, R) = 0$, then $U \subseteq Z$.

**Proof.** (1) Suppose that $d(U) = 0$. Then $d(U, R)_{\sigma, \tau} = 0$. This implies that $U \subseteq C_{\sigma, \tau}$ or $v + \tau\sigma^{-1}(v) \in C_{\sigma, \tau}$ for all $v \in U$ by Theorem 2.

(2) Taking $\alpha = \beta = 1$ in Theorem 2, we have $U \subseteq Z$.

**Theorem 4.** Let $U$ be a nonzero $(\sigma, \tau)$-left Lie ideal of $R$ and $d : R \rightarrow R$ a nonzero $(\alpha, \beta)$-derivation.

1. If $d(U) = 0$, then $\sigma(v) + \tau(v) \in Z$ for all $v \in U$.
2. If $a \in R$ and $[U, a] = 0$, then $a \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in U$. 
(3) If \( a \in R \) and \([U, a]_{\alpha, \beta} = 0\), then \( a \in Z \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \).

(4) If \([ [R, U]_{\alpha, \beta}, a]_{\lambda, \mu} = 0\) then \( a \in Z \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \).

Proof. (1) Suppose that \( d(U) = 0 \). Then \( d[R, v]_{\sigma, \tau} = 0 \) for all \( v \in U \). This implies that \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \) by [4, Corollary 5] for all \( v \in U \).

(2) Let \( d(x) = [x, a] \) for all \( x \in R \). Then \( d \) is a derivation and furthermore \( d(U) = 0 \). Thus we have \( a \in Z \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \) by (1).

(3) Since \([ [R, U]_{\sigma, \tau}, a]_{\alpha, \beta} \subseteq [U, a]_{\alpha, \beta} = 0\) we have \([\tau(U), \beta(a)] = 0\) by Lemma 1. That is \([U, \tau^{-1}\beta(a)] = 0\). This implies that \( a \in Z \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \) by (2).

(4) By Lemma 1 and hypothesis, we have \([\beta(U), \mu(a)] = 0\). That is \([U, \beta^{-1}\mu(a)] = 0\). This implies that \( a \in Z \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \) by (2). \(\square\)

Remark 1. Let \( U \) be a nonzero \((\sigma, \tau)\)-left Lie ideal of \( R \) such that \([U, U]_{\alpha, \beta} = 0\). Then we have \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \).

Proof. By Theorem 4(3) we have \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \). \(\square\)

Theorem 5. Let \( U \) be a nonzero \((\sigma, \tau)\)-left Lie ideal of \( R \) and \( a \in R \).

(1) If \([a, U]_{\alpha, \beta} = 0\), then \( a \in C_{\alpha, \beta} \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \).

(2) If \([a, [R, U]_{\alpha, \beta}]_{\lambda, \mu} = 0\), then \( a \in C_{\lambda, \mu} \) or \( \alpha(v) + \beta(v) \in Z \) for all \( v \in U \).

(3) If \([R, U]_{\alpha, \beta} \subseteq C_{\lambda, \mu}\), then \( R \) is commutative or \( \sigma(v) = \tau(v) \) for all \( v \in U \).

(4) If \( U \subseteq C_{\lambda, \mu} \), then \( \sigma(v) = \tau(v) \) for all \( v \in U \) or \( R \) is commutative.

Proof. (1) Let \( d(x) = [a, x]_{\alpha, \beta} \) for all \( x \in R \). Then \( d \) is \((\alpha, \beta)\)-derivation of \( R \). Since \([a, [R, U]_{\sigma, \tau}]_{\alpha, \beta} \subseteq [a, U]_{\alpha, \beta} = 0\) then we have \( d[R, U]_{\sigma, \tau} = 0\). This implies that \( a \in C_{\alpha, \beta} \) or \( \sigma(v) + \tau(v) \in Z \) for all \( v \in U \) by [4, Corollary 5].

(2) Considering as in the proof (1) we obtain the result.

(3) Suppose that \([R, U]_{\alpha, \beta} \subseteq C_{\lambda, \mu}\). Then we have \([ [R, U]_{\alpha, \beta}, R]_{\lambda, \mu} = 0\).

This gives \([\beta(U), \mu(R)] = 0\) by Lemma 1 and so \( U \subseteq Z \). Thus \([R, U]_{\sigma, \tau} \subseteq U \subseteq Z \) is obtained. For any \( r, s \in R \), \( v \in U \) we have \( 0 = [r, v]_{\sigma, \tau}, s = [r, \sigma(v) - \tau(v)], s = [r, \sigma(v) - \tau(v)], s + [r, s] \) \( (\sigma(v) - \tau(v)) \) which leads to

\[(2.11) [r, s](\sigma(v) - \tau(v)) = 0 \text{ for all } r, s \in R, v \in U.\]

Since \( R \) is prime and \( \sigma(v) - \tau(v) \in Z \) we get

\[(2.12) [r, s] = 0 \text{ for all } r, s \in R \text{ or } \sigma(v) = \tau(v) \text{ for all } v \in U.\]

and so \( R \) is commutative or \( \sigma(v) = \tau(v) \) for all \( v \in U \).

(4) If \( U \subseteq C_{\lambda, \mu} \), then \([R, U]_{\sigma, \tau} \subseteq C_{\lambda, \mu}\). This implies that \( R \) is commutative or \( \sigma(v) = \tau(v) \) for all \( v \in U \) by (3). \(\square\)
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