A Characterization of $\delta$-quasi-Baer Rings

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Abstract

Let $\delta$ be a derivation on $R$. A ring $R$ is called $\delta$-quasi-Baer (resp. quasi-Baer) if the right annihilator of every $\delta$-ideal (resp. ideal) of $R$ is generated by an idempotent of $R$. In this note first we give a positive answer to the question posed in Han et al. [7], then we show that $R$ is $\delta$-quasi-Baer iff the differential polynomial ring $S = R[x; \delta]$ is quasi-Baer iff $S$ is $\delta$-quasi-Baer for every extended derivation $\delta$ on $S$ of $\delta$. This results is a generalization of Han et al. [7], to the case where $R$ is not assumed to be $\delta$-semiprime.
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Throughout this note $R$ denotes an associative ring with unity, $\delta : R \rightarrow R$ is derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x; \delta]$ the skew polynomial ring whose elements are the polynomials $\Sigma_{i=0}^{n}r_i x^i \in R$, $r_i \in R$, where the addition is defined as usual and the multiplication by $xb = bx + \delta(b)$ for any $b \in R$. For a nonempty subset $X$ of a ring $R$, we write $r_R(X) = \{ c \in R | dc = 0 \text{ for any } d \in X \}$ which is called the right annihilator of $X$ in $R$.

Recall from [9] that $R$ is a Baer ring if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. In [9] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. The class of Baer rings includes the von Neumann algebras. In [6] Clark defines a ring to be quasi-Baer if the right annihilator of every ideal is generated, as a right ideal, by an idempotent. Moreover, he shows the left-right symmetry of this condition by proving that $R$ is quasi-Baer if and only if the left annihilator of every left ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Further work on quasi-Baer rings appears in [3, 4, 5, 10, 11]. An ideal $I$ of $R$ is called $\delta$-ideal if $\delta(I) \subseteq I$. $R$ is called $\delta$-quasi-Baer if the right annihilator of every $\delta$-ideal of $R$ is generated by an idempotent of $R$. Clearly each quasi-Baer ring is $\delta$-quasi-Baer. But the converse is not true (see [7] Example). $R$ is said to be reduced if $R$ has no nonzero nilpotent elements. Note that in a reduced ring $R$, $R$ is Baer if and only if $R$ is quasi-Baer.

In [1], Armendariz has shown that if $R$ is reduced, then $R$ is Baer if and only if the polynomial ring $R[x]$ is a Baer ring. Han et al. [7], have generalized this result by showing that if $R$ is $\delta$-semiprime (i.e., for any $\delta$-ideal $I$ of $R$, $I^2 = 0$ implies $I = 0$), then $R$ is a $\delta$-quasi-Baer ring if and only if the
Ore extension $R[x; \delta]$ is a quasi-Baer ring.

Han et al. (2000) posed this question: If $e(x) \in R[x; \delta]$ is a left semicentral idempotent, then does there exists a left semicentral idempotent $e_0 \in R$ such that $e(x)R[x; \delta] = e_0R[x; \delta]$? In this note first we give a positive answer to this question, then we show that $R$ is $\delta$-quasi-Baer if and only if the differential polynomial ring $S = R[x; \delta]$ is quasi-Baer if and only if $S$ is $\delta$-quasi-Baer for every extended derivation $\delta$ on $S$. This results is a generalization of Han et al. [7], to the case where $R$ is not assumed to be $\delta$-semiirprime.

For a ring $R$ with a derivation $\delta$, there exists a derivation on $S = R[x; \delta]$ which extends $\delta$. For example given in [7], consider an inner derivation $\delta$ on $S$ by $x$ defined by $\delta(f(x)) = xf(x) - f(x)x$ for all $f(x) \in S$. Then $\delta(f(x)) = \delta(a_0) + \cdots + \delta(a_n)x^n$ for all $f(x) = a_0 + \cdots + a_n x^n \in S$ and $\delta(r) = \delta(r)$ for all $r \in R$, which means that $\delta$ is an extension of $\delta$. We call such a derivation $\delta$ on $S$ an extended derivation of $\delta$. For each $a \in R$ and nonnegative integer $n$, there exist $t_0, \ldots, t_n \in \mathbb{Z}$ such that $x^n a = \sum_{i=0}^{n} t_i \delta^{n-i}(a)x^i$.

Lemma 1. (Han et al. Lemma 1) Let $R$ be a ring with a derivation $\delta$ and $\delta$ be an extended derivation of $\delta$ on $S = R[x; \delta]$. If $I$ is a $\delta$-ideal of $R$, then $I[x; \delta]$ is $\delta$-ideal of $S$.

Proof. By ([8], Lemma 1.3), $I[x; \delta]$ is an ideal of $S$. Let $f(x) = a_0 + \cdots + a_n x^n \in I[x; \delta]$. For each $i$, $\delta(a_i x^i) = \delta(a_i) x^i + a_i \delta(x^i) = \delta(a_i) x^i + a_i \delta(x^i) \in I[x; \delta]$. Hence $I[x; \delta]$ is a $\delta$-ideal of $S$. \hfill \Box

Now we give a positive answer to the question posed in Han et al. [7].

Theorem 2. Let $I$ be a $\delta$-ideal of $R$ and $S = R[x; \delta]$. If $r_S(I[x; \delta]) = e(x)S$ for some idempotent $e(x) = e_0 + e_1x + \cdots + e_n x^n \in S$, then $r_S(I[x; \delta]) = e_0 S$.

Proof. Since $Ie(x) = 0$, we have $Ie_i = 0$ for each $i = 0, \ldots, n$. Hence $0 = \delta(Ie_i) = \delta(I)e_i + I\delta(e_i)$ for $i = 0, \ldots, n$. Since $I$ is $\delta$-ideal and $Ie_i = 0$, so $I\delta(e_i) = 0$ for each $i = 0, \ldots, n$. By a similar argument we can show that $I\delta^k(e_i) = 0$ for each $i = 0, \ldots, n$ and $k \geq 0$. Hence $\delta^k(e_i) \in r_S(I[x; \delta])$ for each $i = 0, \ldots, n$ and $k \geq 0$. Thus $\delta^k(e_i) = e(x)\delta^k(e_i)$ and that $e_n \delta^k(e_i) = 0$ for each $i = 0, \ldots, n$ and $k \geq 0$. Hence $\delta^k(e_i) = (e_0 + e_1 x + \cdots + e_{n-1} x^{n-1})\delta^k(e_i)$ and that $e_n \delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0$. Continuing in this way, we have $e_j \delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0, j = 1, \ldots, n$. Thus $\delta^k(e_i) = e_0 \delta^k(e_i)$ for each $i \geq 0, k \geq 0$. Therefore $e(x) = e_0 e(x)$ and that $r_S(I[x; \delta]) = e(x)S \subseteq e_0 S$. Since $\delta^k(e_0) \in r_R(I)$, so $e_0 \in r_S(I[x; \delta])$ and that $e_0 S \subseteq r_S(I[x; \delta])$. Therefore $r_S(I[x; \delta]) = e_0 S$. \hfill \Box

Proposition 3. Let $R$ be a $\delta$-quasi-Baer ring. Then $S = R[x; \delta]$ is a quasi-Baer ring.
Proof. Let $J$ be an arbitrary ideal of $S$. Consider the set $J_0$ of leading coefficients of polynomials in $J$. Then $J_0$ is a $\delta$-ideal of $R$. Since $R$ is $\delta$-quasi-Baer, $r_R(J_0) = eR$ for some idempotent $e \in R$. Since $J_0 e = 0$ and $J_0$ is $\delta$-ideal of $R$, we have $J_0 \delta^k(e) = 0$ for each $k \geq 0$. Hence $\delta^k(e) = e\delta^k(e)$ and $eS \subseteq r_S(J_0[x; \delta])$. Clearly $r_S(J_0[x; \delta]) \subseteq eS$. Thus $r_S(J_0[x; \delta]) = eS$. We claim that $r_S(J) = eS$. Let $f(x) = a_0 + \cdots + a_n x^n \in J$. Then $a_n \in J_0$ and that $a_n \delta^k(e) = 0$ for each $k \geq 0$. Hence $f(x)e = (a_0 + \cdots + a_{n-1} x^{n-1})e = \cdots + a_{n-1} e x^{n-1}$. Thus $a_{n-1} e \in J_0$, and $a_{n-1} \delta^k(e) = a_{n-1} e \delta^k(e)$ for each $k \geq 0$. Hence $a_{n-1} e x^{n-1} e = 0$. Continuing in this way, we can show that $a_i x^i e = 0$, for each $i = 0, \cdots , n$. Hence $f(x)e = 0$ and so $eS \subseteq r_S(J)$. Now, let $g(x) = b_0 + \cdots + b_m x^m \in r_S(J)$ and $f(x) = a_0 + \cdots + a_n x^n \in J$. First, we will show that $a_i x^i b_j x^j = 0$, for $i = 0, \cdots , n$, $j = 0, \cdots , m$. Since $f(x)g(x) = 0$, we have $a_n b_m = 0$. Hence $b_m \in r_r(J_0)$. Since $J_0$ is $\delta$-ideal of $R$, $\delta^k(b_m) \in J_0$ for each $k \geq 0$ and that $b_m \in r_S(J_0[x; \delta])$. Thus $b_m = e b_m$ and $a_n x^n b_m x^m = 0$. Since $f(x)e = (a_0 + \cdots + a_n x^n)e = (a_0 + \cdots + a_{n-1} x^{n-1})e$, we have $a_{n-1} e \in J_0$ and $a_{n-1} \delta^k(e) = a_{n-1} e \delta^k(e) = 0$, for each $k \geq 0$. There exist $t_0, \cdots , t_{n-1} \in \mathbb{Z}$ such that, $a_{n-1} x^{n-1} b_m x^m = a_{n-1} x^{n-1} e b_m x^m = a_{n-1} (\sum_{j=0}^{n-1} t_j \delta^{n-j} - 1) e x^j b_m x^m = (\sum_{j=0}^{n-1} t_j a_{n-1} \delta^{n-j} - 1) e x^j b_m x^m$. Hence $a_{n-1} x^{n-1} b_m x^m = 0$. Continuing in this way, we have $a_i x^i b_j x^j = 0$ for each $i, j$. Therefore $b_j \in r_S(J_0[x; \delta]) = eS$, for each $j \geq 0$. Consequently, $g(x) = e g(x)$ and $r_S(J) = eS$. Therefore $S$ is a quasi-Baer ring.

Theorem 4. Let $R$ be a ring and $S = R[x; \delta]$. Then the following are equivalent:

1. $R$ is $\delta$-quasi-Baer;
2. $S$ is quasi-Baer;
3. $S$ is $\delta$-quasi-Baer for every extended derivation $\delta$ on $S$ of $\delta$.

Proof. (1) $\Rightarrow$ (2). It follows from Proposition 3.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Suppose that $R$ is $\delta$-quasi-Baer for every extended derivation $\delta$ on $S$ of $\delta$. Let $I$ be any $\delta$-ideal of $R$. Then by Lemma 1, $I[x; \delta]$ is $\delta$-ideal of $S$. Since $S$ is $\delta$-quasi-Baer, $r_S(I[x; \delta]) = e(x) S$ for some idempotent $e(x) \in S$. Hence $r_S(I[x; \delta]) = e_0 S$ for some idempotent $e_0 \in R$, by Theorem 2. Since $r_R(I) = r_S(I[x; \delta]) \cap R = e_0 R$, $R$ is $\delta$-quasi-Baer.

Acknowledgement. The author thank the referee for his/her helpful suggestions.

References


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(Received July 15, 2006)