Self Maps of Suspension of Sphere Bundles over Spheres

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1. Introduction. Since $\Sigma(S^m \times S^n)$ is homotopy equivalent to $\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}$, there exists a bijection

$$[\Sigma(S^m \vee S^n), Y] \rightarrow [\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}, Y].$$

However this bijection is not always homomorphically with respect to the natural multiplication of two sets. Of course the latter is necessarily abelian for any $Y$. However, the first group is not always abelian. For example, let $p_n : S^m \times S^n \rightarrow S^m$ and $p_m : S^m \times S^n \rightarrow S^m$ be the projections onto each factor and let $q : S^m \times S^n \rightarrow S^{m+n}$ be the projection. For brevity, suppose $i$ denotes canonical inclusions $\Sigma S^m \rightarrow \Sigma(S^m \times S^n)$, $\Sigma S^n \rightarrow \Sigma(S^m \times S^n)$, $\Sigma S^m \vee \Sigma S^n \rightarrow \Sigma(S^m \times S^n)$ by the same symbol. Then the commutator $< i \circ \Sigma p_n, i \circ \Sigma p_m > = \pm i_*[\iota_{n+1}, \iota_{m+1}] \circ \Sigma q \in [\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is a non-trivial element (Corollary 2.2) and so $[\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is non-abelian. More generally we show,

**Theorem.** Let $E(\xi)$ be an $S^m$-bundle over $S^n$ with its characteristic class $\xi \in \pi_{n-1}(SO(m+1))$. If $2 < m + 1 < n$, then the group $[\Sigma E(\xi), \Sigma E(\xi)]$ is not abelian.

For example the cases of $S^3 \rightarrow Sp(2) \rightarrow S^7$ and $S^3 \rightarrow SU(3) \rightarrow S^5$ are known by Ohshima.

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2. A commutator in $[\Sigma E(\xi), \Sigma E(\xi)]$ We use the notations:

$p_n : S^m \times S^n \rightarrow S^m$ and $p_m : S^m \times S^n \rightarrow S^m$ be the projections on the each of factors, respectively,

$q : S^m \times S^n \rightarrow S^{m+n} = S^m \times S^n / S^m \vee S^n$ be the projection,

$i_n : S^n \subset \Omega \Sigma(S^m \vee S^n)$ and $i_m : S^m \subset \Omega \Sigma(S^m \vee S^n)$ be the canonical inclusions respectively,

$i$ denotes the adjoints of $i_n$ and $i_m$ by the same symbol,

$+, -$ denotes the loop sum operation and its inverse,

and assume $2 < m + 1 < n$.

**Lemma 2.1.** In $[\Sigma(S^m \times S^n), \Sigma(S^m \vee S^n)]$, we have the following
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\[ i \circ \Sigma p_n, i \circ \Sigma p_m \] = [\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.

where \( \{a, b\} \) denotes the commutator \((-a - b) + (a + b)\).

**Proof.** The adjoint of \( \{i \circ \Sigma p_n, i \circ \Sigma p_m\} \) is represented \( \langle i_n, i_m \rangle \circ q \) using the Samelson product \( \langle i_n, i_m \rangle \) of \( i_n \) and \( i_m \). Considering the adjoint by [3], this turns to the identity

\[ i \circ \Sigma p_n, i \circ \Sigma p_m \] = \pm [\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.

Applying the inclusion \( \Sigma(S^m \vee S^n) \subset \Sigma(S^m \times S^n) \), the following is easy;

**Corollary 2.2.** \( \Sigma(S^m \times S^n) \), \( \Sigma(S^m \times S^n) \) is non-abelian.

Next we consider more generally the case \( \Sigma E(\xi), \Sigma E(\xi) \) where

\( p : E(\xi) \to S^n \), an \( S^m \)-bundle over \( S^n \) with its characteristic class \( \xi \in \pi_{n-1}(SO(m+1)) \),

\( q : E(\xi) \to E(\xi)/S^m \cup e^n = S^{m+n} \).

We assume \( 2 < m+1 < n \). The following lemma is an extension of [2].

**Lemma 2.3.** Let \( X \) be a connected finite CW complex of \( \text{dim } X = n \) and let \( P : X \to X/X_{n-1} = \vee_k S^n \) be the projection where \( X_{n-1} \) denotes the \((n-1)\)-skelton of \( X \). Then for \( f \in [\Sigma X, Y] \) and \( \{\alpha_k\} \in \pi_{n+1}(Y) \), we have the equality,

\[ f + (\vee_k \alpha_k) \circ \Sigma P = (\vee_k \alpha_k) \circ \Sigma P + f \]

in the group \([\Sigma X, Y]\).

**Proof.** We put \( S = \Sigma(X/X_{n-1}) = \vee_k S^{n+1} \) and consider the map \( \Sigma P + \Sigma id : \Sigma X \to S \vee \Sigma X \). Since the inclusion \( i : S \vee \Sigma X \to S \times \Sigma X \) induces a bijection \([\Sigma X, S \vee \Sigma X] \to [\Sigma X, S \times \Sigma X] \) because of \((S \times \Sigma X)_{n+2} = S \vee \Sigma X \) we obtain \( \Sigma P + \Sigma id = \Sigma id + \Sigma P \). Then the proof is completed by applying the map \((\vee_k \alpha_k) \vee f : S \vee \Sigma X \to Y \) to both sides of this equality.

**Lemma 2.4.** Let \((X, \mu)\) be a connected CW Hopf space, \( n > m+1 > 2 \), \( \alpha \in \pi_n(X) \) and \( g : E(\xi) \to X \). Then

\[ \{\alpha \circ p, g\}_\mu = \pm < \alpha, g \circ i >_\mu \circ q, \]

where \( \{\alpha, \beta\}_\mu = (-\alpha - \beta) + (\alpha + \beta) \) is the commutator in the algebraic loop \([E(\xi), X]\) with respect to \( \mu \), \( < -, - >_\mu \) is the Samelson product with
respect to $\mu$.

**Proof.** It follows from the commutativity of the following diagram.

\[
\begin{array}{cccc}
E(\xi) & \rightarrow & S^n \wedge S^m & \rightarrow & X \\
\downarrow d & & \downarrow id \wedge i & & \uparrow \{ \}_\mu \\
E(\xi) \wedge E(\xi) & \rightarrow & S^n \wedge E(\xi) & \rightarrow & X \wedge X \\
(\epsilon \circ p) \wedge id & & \alpha \wedge g & &
\end{array}
\]

where $d$ denotes the diagonal map, $\{ \}_\mu$ denotes the commutator map with respect to $\mu$ and $\epsilon$ is $\pm 1$. The commutativity of the first square follows from the facts that $(id \wedge i) \circ q$ and $\{(\epsilon \circ p) \wedge id\} \circ d$ have the same induced homomorphism $H^{m+n}(S^n \wedge E(\xi)) \rightarrow H^{m+n}(E(\xi))$ and the natural transformation

\[ [E(\xi), S^n \wedge E(\xi)] \rightarrow \text{Hom}(H^{m+n}(S^n \wedge E(\xi)), H^{m+n}(E(\xi))). \]

is isomorphic. The commutativity of the second square follows from the definition. Thus the proof is completed.

**Proposition 2.5.** If $\alpha \in [\Sigma S^n, \Sigma E(\xi)]$ and $g \in [\Sigma E(\xi), \Sigma E(\xi)]$, then we have

\[ \{\alpha \circ \Sigma p, g\}_\mu = (-1)^n \epsilon[\alpha, g \circ \Sigma i] \circ \Sigma q. \]

**Proof.** It follows from the adjoint isomorphism

\[ [\Sigma E(\xi), \Sigma E(\xi)] \simeq [E(\xi), \Omega \Sigma E(\xi)] \]

\[ (\Sigma q)^* \uparrow \quad \uparrow q^* \]

\[ \pi_{m+n+1}(\Sigma E(\xi)) \simeq \pi_{m+n}(\Omega \Sigma E(\xi)) \]

and Lemma 2.4.

**Proof of Theorem.** From the exact sequence of homotopy groups of the fiber bundle $p : E(\xi) \rightarrow S^n$, there exists an element $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ such that $\text{deg}(\Sigma p \circ \alpha)$ is non-zero, because $\pi_{n-1}(S^m)$ is finite. By [1], $\Sigma E(\xi)$ has the homotopy type of the mapping cone of $\Sigma \Delta_{i_n} \vee J(\xi) : \Sigma S^{n-1} \vee \Sigma S^{m+n-1} \rightarrow \Sigma S^m$ where $\Delta$ denotes the boundary homomorphism $\pi_n(S^n) \rightarrow \pi_{n-1}(S^m)$ of the exact sequence of homotopy groups of the fiber bundle and $J$ is the Hopf-Whitehead J-homomorphism. By our assumption $n > m+1$, $J(\xi) \in \pi_{n+m}(\Sigma S^n)$ has the finite order. For such an element $\alpha$, it follows that $[\alpha, \Sigma i]$ has infinite order because there
exists no map $S^{n+1} \times S^{m+1} \to \Sigma E(\xi)$ of type $(k\alpha, \Sigma i)$ for any integer $k \neq 0$.

From the Puppe sequence

$$\Sigma(\Sigma i \circ J(\xi))^* \quad \Sigma q^*$$

$$[\Sigma^2(S^m \cup e^n), \Sigma E(\xi)] \quad \to \quad \pi_{m+n+1}(\Sigma E(\xi)) \quad \to \quad [\Sigma E(\xi), \Sigma E(\xi)]$$

it follows that the order of the kernel of $\Sigma q^* : \pi_{m+n+1}(\Sigma E(\xi)) \to [\Sigma E(\xi), \Sigma E(\xi)]$ is finite and so we have that the commutator $<\alpha \circ \Sigma p, id_{\Sigma E}>$ is also non-trivial by Proposition 2.5. Thus the proof is completed.

**Remark.** (1) Any maps in $[\Sigma E(\xi), \Sigma E(\xi)]$ can be represented as the formula

$$s \ id_{\Sigma E} + \alpha \circ \Sigma p + \beta \circ \Sigma q$$

for $s \in Z$ (integers), $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ and $\beta \in \pi_{m+n}(\Sigma E(\xi))$.

(2) On the set $[\Sigma E(\xi), \Sigma E(\xi)]$, the iterated commutators are trivial.

(3) For the case $n = m+1$, we have a countereexample for the Theorem, that is, the Hopf fibration $S^3 \to S^7 \to S^4$ or $S^7 \to S^{15} \to S^8$.

**References**


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