On conformal collineations

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ON CONFORMAL COLLINEATIONS

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This paper is devoted to the study of the decomposition of a conformal collineation relative to the reducibility of a manifold.

§ 1. Conformal collineation on an irreducible Riemannian manifold.

We consider an $n$-dimensional Riemannian manifold $M$ with metric tensor $g_{\mu\lambda}$. The Christoffel symbol, the curvature tensor and the Ricci tensor are denoted by $\{s_{\mu\lambda}\}$, $K_{\gamma\mu\lambda} \xi^\gamma$ and $K_{\mu\lambda}$ respectively.

An infinitesimal transformation $v^\gamma$ is called a conformal collineation if it satisfies the equation

\[(1.1) \quad \xi^\gamma_{\xi} s_{\mu\lambda} = \Gamma^\gamma_{\mu\lambda} v^\gamma + v^\gamma K_{\gamma\mu\lambda} = A^\gamma_{\mu\lambda} \sigma^\lambda + A^\gamma_{\lambda} \sigma^\mu - g_{\mu\lambda} \sigma^\gamma,
\]

where $\xi^\gamma$ indicates the Lie differentiation with respect to $v^\gamma$, $\Gamma^\gamma_{\mu\lambda}$ the covariant differentiation, $A^\gamma_{\mu\lambda}$ is the unity tensor and $\sigma^\mu$ is a vector field. The class of conformal collineations contains affine and conformal transformations. Since we have

\[(1.2) \quad \Gamma^\gamma_{\mu\lambda} v^\gamma = n \sigma^\gamma,
\]

$\sigma^\gamma$ is the gradient vector field of a scalar function $\sigma$.

\[(1.3) \quad \sigma^\gamma = \partial_\lambda \sigma^\gamma.
\]

Substituting (1.1) into the well-known formula [5, p. 17]

\[(1.4) \quad \xi^\gamma K_{\gamma\mu\lambda} = \Gamma^\gamma_{\gamma\mu\lambda} - \Gamma^\gamma_{\mu\lambda} \xi^\gamma_{\xi},
\]

we obtain the equation

\[(1.5) \quad (\xi^\gamma K_{\gamma\mu\lambda} s_{\mu\lambda}) v^\gamma_{\xi} = v^\gamma \Gamma^\gamma_{\gamma\mu\lambda} + K_{\gamma\mu\lambda\xi} \Gamma^\gamma_{\gamma\mu\lambda} v^\gamma + K_{\gamma\mu\lambda\xi} \Gamma^\gamma_{\mu\lambda} v^\gamma
\]

\[+ K_{\gamma\mu\lambda\xi} \Gamma^\gamma_{\gamma\mu\lambda} v^\gamma - K_{\gamma\mu\lambda\xi} \Gamma^\gamma_{\mu\lambda} v^\gamma
\]

\[= -g_{\mu\lambda} \Gamma^\gamma_{\mu\lambda} \sigma^\gamma + g_{\mu\lambda} \Gamma^\gamma_{\mu\lambda} \sigma^\gamma + g_{\mu\lambda} \Gamma^\gamma_{\mu\lambda} \sigma^\gamma.
\]

Now, from (1.1), we have

\[(1.6) \quad \Gamma^\gamma_{\mu\lambda} (\Gamma^\gamma_{\mu\lambda} v^\gamma + \Gamma^\gamma_{\mu\lambda} v^\gamma) = 2\sigma^\gamma g_{\mu\lambda},
\]

or

\[(1.7) \quad \Gamma^\gamma_{\mu\lambda} (\Gamma^\gamma_{\mu\lambda} v^\gamma + \Gamma^\gamma_{\mu\lambda} v^\gamma - 2\sigma^\gamma g_{\mu\lambda}) = 0.
\]

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1) All transformations appearing in this paper are infinitesimal, so we shall omit the modifier "infinitesimal".
If the Riemannian manifold $M$ is irreducible, we have therefore
\[(1.8)\]
$$\Gamma^\lambda_{\kappa\nu} + \Gamma^\nu_{\kappa\lambda} - 2\sigma g_{\lambda\nu} = 2c g_{\lambda\nu},$$
c being a constant. Thus the vector field $v^\rho$ satisfies the equation
\[(1.9)\]
$$\xi_\rho g_{\mu\lambda} = 2(\sigma + c)g_{\mu\lambda},$$
and we obtain the following

**Theorem 1.** If a Riemannian manifold $M$ is irreducible, then a conformal collineation on $M$ is a conformal transformation.

§ 2. Conformal collineation on a locally reducible Riemannian manifold.

Let a Riemannian manifold $M$ be locally a product
\[(2.1)\]
$$M_0 \times M_1 \times \cdots \times M_r,$$
where $M_0$ is the euclidean part and $M_1, \ldots, M_r$ are the irreducible parts.
Let each part $M_t$ be of dimension $n_t$ ($t = 0, 1, \ldots, r$); $n_0 + n_1 + \cdots + n_r = n$. There exists then a local coordinate system $(x^0, x^1, \ldots, x^n)$, called a separated coordinate system, where the metric tensor field $g_{\mu\lambda}$ is given by a reduced matrix
\[(2.2)\]
$$\begin{pmatrix}
\delta_{j0}\delta_{k0} & 0 & & \\
& \ddots & & \\
& & \ddots & \\
0 & & & \delta_{jr}\delta_{kr}
\end{pmatrix},$$
$\delta_{jk}$ being the Kronecker delta and the notation $^*$ meaning that the equation holds in a separated coordinate system. In such a system, the non-vanishing components of $\{g_{\mu\lambda}\}$ and $K_{\nu\mu\lambda}$ are only $\{g_{ij}\}$ and $K_{ijij}$ respectively, which are dependent only of the variables $x^t$ belonging to $M_t$ ($t = 1, 2, \ldots, r$). If we define tensor fields $\xi^i g_{\mu\lambda}$ ($t = 0, 1, \ldots, r$) by
\[(2.3)\]
$$\xi^i g_{\mu\lambda} = 0$$
and
\[(2.4)\]
$$g_{\mu\lambda} = g_{\mu\lambda}^0 + g_{\mu\lambda}^1 + \cdots + g_{\mu\lambda}^r,$$
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Now, referring the equation (1.5) to a separated coordinate system and putting $\kappa = h_s$, $\lambda = i_t$, $\mu = j_t$, $\nu = k_s$ ($s \neq t$), we have

$$-g_{k_s h_s} \Gamma_{k_s}^l \sigma_l - g_{j_t i_t} \Gamma_{j_t}^l \sigma_l = 0$$

and consequently

$$\Gamma_{j_t}^l \sigma_l = \alpha_t g_{j_t i_t} \quad (t = 0, 1, \cdots, r)$$

and the proportional factors $\alpha_0, \alpha_1, \cdots, \alpha_r$ satisfy the relations

$$\alpha_s + \alpha_t = 0 \quad (s \neq t).$$

If $M$ has at least three parts, then the proportional factors $\alpha_t$ all vanish and we have

$$\Gamma_{j_t}^l \sigma_l = 0 \quad (t = 0, 1, \cdots, r).$$

Moreover, putting $\kappa = h_u$, $\lambda = i_t$, $\mu = j_t$, $\nu = k_u$ ($s$, $t$, $u \neq t$) in (1.5), we have also

$$\Gamma_{j_t}^l \sigma_l = 0 \quad (s \neq t).$$

The equations (2.8) and (2.9) together make up the tensor equation

$$\Gamma_{\mu}^l \sigma_\lambda = 0.$$

The equations (2.9) imply that $\sigma$ may be written in the form

$$\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_r,$$

where each $\sigma_t$ is a function depending only on the variables $x^u$ belonging to $M_t$ in a separated coordinate system. However, by (2.8), $\partial_t \sigma_t$ is a parallel vector field on the part $M_t$ and hence $\sigma_t$ for $t = 1, \cdots, r$ are constants in virtue of the irreducibility of $M_t$ and $\sigma_0$ is a linear function of the variables $x^u$ belonging to the euclidean part $M_0$. Thus $\sigma$ may be written as

$$\sigma = a_{i_0} x^{i_0} + \alpha,$$

$a_{i_0}$ and $\alpha$ being constants.

On the other hand, putting $\kappa = h_s$, $\lambda = i_t$, $\mu = j_t$ ($s$, $t$, $u \neq t$) in (1.6), we have

$$\partial_{j_u} (\Gamma_{i_t}^{v_s} u_{h_s} + \Gamma_{h_s}^v p_t) = 0,$$

and therefore the expressions in the parentheses are dependent only of $x^u$ and $x^v$. Putting also $\kappa = h_s$, $\lambda = i_t$, $\mu = j_t$ ($s \neq t$), we have

$$\Gamma_{j_s} (\Gamma_{i_t}^{v_s} u_{h_s} + \Gamma_{h_s}^v p_t) = 0.$$
we have hence

\[(2.15) \quad \Gamma^s_{ti} v^h_s + F^h_s v^s_t = 0\]

for any pair of \(h_s\) and \(i_s\) \((s \neq t)\). Moreover, from (1.1), we have

\[(2.16) \quad \partial_j \Gamma^j_{ti} v^h_t = \sigma_j \partial^h_t,\]

and consequently the equations

\[(2.17) \quad \Gamma^j_{ti} v^h_t = \sigma g_{ij} h_s + f^i_{h_s},\]

where \(f^i_{h_s}\) are functions dependent only of \(x^a\). Substituting (2.17) into (1.6) referred to \(M_n\), we have

\[(2.18) \quad \Gamma^j_{ti} (f^i_{h_s} + f^i_{h_s}) = 0.\]

Therefore we see that for \(s = 0\)

\[(2.19) \quad 2 \beta_{i_0} = f^i_{h_0} + f^i_{h_0},\]

are constants and for \(s \neq 0\),

\[(2.20) \quad f^i_{h_0} + f^i_{h_0} = 2c^i g^i_{h_0},\]

c, being constants. Thus we have

\[(2.21) \quad \Gamma^j_{ti} v^h_t \equiv 2 \sigma g_{ij} h_s + 2 \beta_{i_0} h^s_t\]

and

\[(2.22) \quad \Gamma^j_{ti} v^h_t + \Gamma^h_s v^s_t = 2 \sigma g_{ij} h_s + 2c^i g^i_{h_0},\]

\((s \neq 0)\).

If we define a tensor field \(\beta_{i_0}\) by

\[(2.23) \quad \beta_{i_0} = \begin{pmatrix} \beta_{i_0} \varepsilon^i_0 \\ 0 \end{pmatrix},\]

then \(\beta_{i_0}\) is a symmetric parallel tensor field. The equations (2.15), (2.21) and (2.22) together make up the tensor equation

\[(2.24) \quad \xi_i g^i_{\mu \lambda} = 2 \sigma g_{\mu \lambda} + 2 \beta_{i_0} + 2 \sum \epsilon_i g^i_{\mu \lambda}.\]

Conversely, if a vector field \(\nu^i\) satisfies the equation (2.24), then we substitute (2.24) into the well-known equation

\[(2.25) \quad \xi_i g^i_{\mu \lambda} = \frac{1}{2} g^e_{\alpha \beta} (\Gamma^e_{\mu \alpha} g_{\lambda \beta} + \Gamma^e_{\lambda \beta} g_{\mu \alpha} - \Gamma^e_{\alpha \beta} g_{\mu \lambda}),\]

and obtain the equation (1.1). Thus we have established

**Theorem 3.** In order that a vector field \(\nu^i\) be a conformal collineation, it is necessary and sufficient that \(\nu^i\) satisfy the equation (2.24).

From (2.6) and (2.22), we notice here that the vector field given by \(\nu^i\) on each irreducible part \(M_i\), which we call the restriction of \(\nu^i\) on \(M_i\), defines a concircular transformation [7].
§ 3. Conformal collineation in a locally euclidean manifold.

A locally euclidean manifold $M$ of dimension $n \geq 2$ may be regarded locally as a product of $n$ straight lines. Accordingly, in a local orthogonal coordinate system $(x^i)$, the function $\sigma$ is given by

$$(3.1) \quad \sigma = \sum a_i x_i + a.$$  

The equation (1.1) is reduced to

$$(3.2) \quad \partial_j \partial_i v^h = \delta_{ih} a_j + \delta_{jh} a_i - \delta_{ij} a_h.$$  

We seek for the general solution of this equation, cf. [3]. First, from (3.2) with $h, i, j \neq i$, we see that $\partial_i v^h$ are dependent only of the variables $x^i$ and $x_h$. If $h \neq i$ in (3.2), we have

$$(3.3) \quad \partial_i \partial_i v^h = -a_h \quad (h \neq i),$$  

from which

$$(3.4) \quad \partial_i v^h = -a_h x_i + \phi_{ih} \quad (h \neq i),$$  

$\phi_{ih}$ being a function of $x_h$. For $h = j \neq i$ in (3.2), we have

$$(3.5) \quad \partial_h \partial_i v^h = \frac{d \phi_{ih}}{dx_h} = a_i \quad (h \neq i)$$  

and hence

$$(3.6) \quad \phi_{ih} = a_i x_h + b_{ih} \quad (h \neq i),$$  

$b_{ih}$ being constants. Therefore, from (3.4), we see that the components $v^h$ are written in the form

$$(3.7) \quad v^h = -\frac{1}{2} a_h \sum x_i^2 + x_h \sum a_i x_i + \sum b_{ih} x_i + \psi_h,$$  

where, for each value of $h$, $\psi_h$ is a function of $x_h$. From (3.2) we have also

$$(3.8) \quad \partial_h \partial_h v^h = a_h$$  

and, substituting (3.7) into these equations,

$$(3.9) \quad \frac{d^2 \psi_h}{dx_h^2} = a_h,$$  

from which

$$(3.10) \quad \psi_h = \frac{1}{2} a_h x_h^2 + b_{hh} x_h + b_h,$$  

$b_{hh}$ and $b_h$ being constants. Thus the vector field $v^h$ is expressed as

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2) In this paragraph we do not adopt the summation convention and omit the notation $^\bullet$ for equations in an orthogonal coordinate system.
(3.11) \[ v^h = -\frac{1}{2} a_h \sum_i x_i^2 + x_h \sum_i a_i x_i + \sum_i b_{ih} x_i + b_h. \]

If we define vector fields \( u^h \) and \( w^h \) by

(3.12) \[ u^h = \sum_i b_{ih} x_i + b_h, \]

(3.13) \[ w^h = -\frac{1}{2} a_h \sum_i x_i^2 + x_h \sum_i a_i x_i, \]

then \( u^h \) defines an affine transformation and \( w^h \) a conformal transformation in the locally euclidean manifold. Thus we have

**Theorem 3.** A conformal collineation \( v^e \) in a locally euclidean manifold is decomposed into

(3.14) \[ v^e = u^e + w^e, \]

where \( u^e \) is an affine transformation and \( w^e \) a conformal transformation. As it can be easily proved, the decomposition (3.14) is unique to within a homothetic transformation.

Since the conformal homeomorphism of a euclidean space onto itself is only a homothety, we can obtain

**Theorem 4.** If a conformal collineation \( v^e \) on a euclidean space generates a global one-parameter group of transformations, then the collineation is affine.

§ 4. The case where \( M \) has at least three parts.

By means of the notice at the beginning of § 3, the case where the euclidean part is of dimension \( \geq 2 \) is one of the present cases.

If no part of \( M \) is locally euclidean in this case, then, by the argument preceding (2.12), the function \( \sigma \) is constant and we have

(4.1) \[ \mathcal{L}_v \{ e_{\mu \lambda} \} = 0, \]

that is

**Theorem 5.** If a Riemannian manifold \( M \) has at least three parts and no part is locally euclidean, then a conformal collineation on \( M \) is an affine transformation.

By use of a theorem due to S. Ishihara and M. Obata [1] and S. Kobayashi [2], we can further say

**Theorem 6.** If, in addition to the assumption of the above theorem, the manifold \( M \) is complete, then a conformal collineation on \( M \) is an isometry.

If there exists a euclidean part \( M_0 \), then \( \sigma \) is given by (2.12) and we have

(4.2) \[ \sigma_{v^e} \uparrow \frac{e}{\sigma_{v^e}}, \quad \sigma_{z^e} \uparrow 0 \quad (s \neq 0). \]
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The equation (1.1) with \( \kappa = h_0 \) is separated into the following equations:

\[
\Gamma_{i_4} \Gamma_{i_3} v_{h_0} \equiv -g_{i_4} a_{h_0}
\]

\[(4.3)\]
\[
\partial_{i_4} \partial_{i_3} v_{h_0} \equiv 0, \quad (s, t \neq 0)
\]

\[
\partial_{i_4} \partial_{i_3} v_{h_0} \equiv \delta_{i_4} a_{i_0} + \delta_{i_4} a_{i_0} - \delta_{i_4} a_{h_0}
\]

By the second equations \( \partial_{i_4} v_{h_3} \) are independent of \( x^a (s \neq 0) \), and by the third equations we have the expressions

\[(4.4)\]
\[
v_{h_0} \equiv -\frac{1}{2} a_{h_0} \sum x^2_i + x_{h_0} \sum a_{i_0} x_i + \sum b_{i_0} x_i + \gamma_{h_0}
\]

\( \gamma_{h_0} \) being the functions independent of \( x^a \). Substituting (4.4) into the first of (4.3), the functions \( \gamma_{h_0} \) are solutions of the equations

\[(4.5)\]
\[
\Gamma_{i_4} \Gamma_{i_3} \gamma_{h_0} \equiv -g_{i_4} a_{h_0} \quad (s, t \neq 0).
\]

Now we define a vector field \( w^i \) by the equations

\[(4.6)\]
\[
w_{h_0} \equiv -\frac{1}{2} a_{h_0} \sum x^2_i + x_{h_0} \sum a_{i_0} x_i + \gamma_{h_0} \quad (s \neq 0)
\]

\[
w_{i_4} \equiv -\sum x_{h_0} \partial_{i_4} \gamma_{h_0}
\]

in the separated system. We can easily verify that the vector field \( w^i \) satisfies the equation

\[(4.7)\]
\[
\xi_{\mu} g_{\mu\lambda} = 2a g_{\mu\lambda},
\]

that is, \( w^i \) is a conformal transformation. Since the equation (1.1) holds also for \( w^i \), if we put

\[(4.8)\]
\[
u^i = \nu^i - w^i,
\]

then we have

\[(4.9)\]
\[
\xi_{\mu} \{\lambda\} = 0,
\]

that is, the vector field \( \nu^i \) is an affine transformation. Thus

**Theorem 7.** If a locally reducible Riemannian manifold \( M \) has at least three parts, one of which is euclidean, then a conformal collineation \( \nu^i \) on \( M \) is decomposed into

\[(4.10)\]
\[
\nu^i = \nu^i + w^i,
\]

where \( \nu^i \) is an affine transformation and \( w^i \) a conformal transformation.

Since, in the present case, the function \( \sigma \) depends only on the points of \( M \), the equations (2.22) means that the restriction of \( \nu^i \) on each part \( M_s (s \neq 0) \) defines a homothetic transformation on \( M_s \). If \( M \) is complete and simply connected, then \( M_s (s \neq 0) \) are complete, simply connected and irreducible. By means of a well-known theorem [1], the homothetic trans-
formation should be an isometry on each $M_s (s \neq 0)$. Hence

$$c_s = -\sigma$$

and $\sigma$ is constant. Then the equation (2.24) is reduced to

$$\xi^\mu g_{\mu\lambda} = 2\sigma g_{\mu\lambda} + 2\beta_{\mu\lambda},$$

and the collineation is affine. The simple connectedness can be removed and we obtain the following

**Theorem 8.** If, in addition to the assumption of Theorem 7, the manifold $M$ is complete, then a conformal collineation on $M$ is an affine transformation.

§ 5. The case where $M$ has two irreducible parts.

We can not go on with the discussions in this general case as yet, but proceed in the case of a manifold of constant scalar curvature, to which we shall confine ourselves in this paragraph. We call here $\hat{K} = K_{\mu\lambda}g^{\mu\lambda}$ and $k = K/n(n - 1)$ the contracted curvature and the scalar curvature of an $n$-dimensional manifold $M$ respectively.

Let $M$ be locally the product of two parts:

$$(5.1) \quad M = M_1 \times M_2.$$  

There occur the two following cases:

i) The two parts are both irreducible.

ii) One part is irreducible and the other is a straight line.

First we consider Case i). Denote the contracted and scalar curvatures of the part $M_s$ by $K_s$ and $k_s$:

$$(5.2) \quad K_s = K_{s\mu\lambda}g^{s\mu\lambda}, \quad k_s = \frac{K_s}{n_{s}(n_{s} - 1)} \quad (s = 1, 2).$$

We have clearly

$$(5.3) \quad K = K_1 + K_2$$

and $K_1$ and $K_2$ are constant, and consequently so are $k_1$ and $k_2$. Since the restrictions on $M_1$ and $M_2$, denoted here by $v_1$ and $v_2$, of a conformal collineation $v^c$ define concircular transformations on $M_1$ and $M_2$ respectively, we can derive the equations

$$(5.4) \quad \xi^\mu K_1 = -2(\sigma + c_1)K_1 - 2(n_1 - 1)n_1\alpha_1 = 0,$$

$$(5.4) \quad \xi^\mu K_2 = -2(\sigma + c_2)K_2 - 2(n_2 - 1)n_2\alpha_2 = 0$$

from the equations similar to (1.5) for the restrictions $v_1$ and $v_2$ by taking account of (2.6) and (2.22). By (2.7) we may put

$$(5.5) \quad \alpha_1 = -\alpha_2 = -\alpha$$
and then from (5.4) follow the equations

\begin{equation}
\alpha = (\sigma + c_1)k_1 = -(\sigma + c_2)k_2
\end{equation}

or

\begin{equation}
(k_1 + k_2)\sigma = -(c_1k_1 + c_2k_2).
\end{equation}

If \(k_1 + k_2 \neq 0\), we see that \(\sigma\) is a constant and the collineation is affine. If \(k_1 = -k_2 \neq 0\), we have \(c_1 = c_2\) and the collineation is a conformal transformation. If \(k_1 = k_2 = 0\), then \(\alpha\) vanishes identically and we have

\begin{equation}
\Gamma_{j_1}^{*}\sigma_{j_1} = \Gamma_{j_2}^{*}\sigma_{j_2} = 0.
\end{equation}

In virtue of the irreducibility of \(M_1\) and \(M_2\), we have \(\sigma = 0\) and the collineation is affine. Combining these results with Theorem 5, we obtain the following

**Theorem 9.** Let a Riemannian manifold \(M\) be of constant scalar curvature and have no euclidean part. If \(M\) itself is irreducible or \(M\) is the product of two irreducible parts whose scalar curvatures are signed oppositely to each other, then a conformal collineation on \(M\) is a conformal transformation. Otherwise it is an affine transformation.

Next we consider Case ii). We suppose that in (5.1) \(M_1\) is the irreducible part and \(M_2\) the straight line. Then the indices belonging to \(M_2\) take only the number \(n\). Clearly \(K_1\) satisfies the first equation of (5.4), and \(K_2\) and \(k_2\) vanish. Thus we have

\begin{equation}
\alpha = (\sigma + c)k_1
\end{equation}

and, from (2.6), (2.7) and (2.22), the equations

\begin{equation}
\Gamma_{j_1}^{*}\sigma_{j_1} = -(\sigma + c_1)k_1g_{j_1j_1},
\end{equation}

\begin{equation}
\Gamma_{n}^{*}\sigma_{n} = (\sigma + c_1)k_1.
\end{equation}

If we define a vector field \(w\) by

\begin{equation}
w_{j_1} = -\sigma_{j_1},
\end{equation}

\begin{equation}
w_{n} = \sigma_n,
\end{equation}

then it is verified that the vector field \(w\) satisfies the equation

\begin{equation}
\mathcal{L}_w g_{\mu\lambda} = \Gamma_{\mu}^{*}w_{\lambda} + \Gamma_{\lambda}^{*}w_{\mu} = 2(\sigma + c_1)k_1g_{\mu\lambda}.
\end{equation}

Hence \(w\) is a conformal transformation. On the other hand, putting

\begin{equation}
c_2 = \tilde{\beta}_{nn},
\end{equation}

the equation (2.24) is written as
(5.14) \[ \mathcal{L}_v \bar{g}_{\mu \lambda} = 2(\sigma \bar{g}_{\mu \lambda} + c_1 \bar{g}_{\mu \lambda} + c_2 \bar{g}_{\mu \lambda}). \]

If \( k_1 \neq 0 \) and we put

(5.15) \[ u^s = v^s - \frac{1}{k_1} \omega^s, \]

then, from (5.12) and (5.15), we have

(5.16) \[ \mathcal{L}_u \bar{g}_{\mu \lambda} = 2(c_2 - c_1) \bar{g}_{\mu \lambda}. \]

Substituting (5.16) into (2.25), we can see that the vector field \( u^s \) defines an affine transformation.

If \( k_1 = 0 \), then we have

(5.17) \[ \Gamma_{\lambda}^{\mu} \sigma_{\lambda} = \Gamma_{\lambda}^{\mu} \sigma_{\lambda} = 0 \]

and, by the irreducibility of \( M_1 \),

(5.18) \[ \sigma_{\lambda} = 0, \quad \sigma_{\lambda} = a_n \]

and hence

(5.19) \[ \sigma = a_n x^a + a, \]

where \( a_n \) and \( a \) are constants. By the same argument as that in § 4, the \( n \)-th component of \( v^s \) is given by

(5.20) \[ v^n = \frac{1}{2} a_n x_n^2 + c_2 x_n + \gamma, \]

where \( \gamma \) is a function of the variables \( x^a \) belonging to \( M_1 \) and satisfies the equation

(5.21) \[ \Gamma_{\lambda}^{\mu} \Gamma_{\mu}^{\nu} \gamma = -g_{\mu \lambda} a_n. \]

If we define a vector field \( w^s \) by the equations

(5.22) \[ w_{\lambda} = -x_{\lambda} \partial_{\lambda} \gamma, \quad w_n = \frac{1}{2} a_n x_n^2 + \gamma, \]

in the separated coordinate system, then the vector field \( w^s \) is a conformal transformation satisfying the equation

(5.23) \[ \mathcal{L}_w g_{\mu \lambda} = 2\sigma g_{\mu \lambda}. \]

Moreover we can see that the vector field \( u^s \) given by

(2.24) \[ u^s = v^s - w^s \]

is an affine transformation. Combining these results with Theorem 7, we establish the following

**Theorem 10.** Suppose that a Riemannian manifold \( M \) is of constant
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scalar curvature and has a euclidean part. Then a conformal collineation \( v^e \) on \( M \) is decomposed into

\[
(5.25) \quad v^e = u^e + w^e,
\]

where \( u^e \) is an affine transformation and \( w^e \) a conformal one.

Thus the further discussions on conformal collineations, in particular, of a complete and reducible Riemannian manifold, are connected with K. Yano and T. Nagano's study [6] as for the part of affine transformation and with the author's recent work [4] as for the part of conformal transformation.

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