Engineering

Industrial & Management Engineering fields

Okayama University *Year* 2003

An optimal finite-dimensional modeling in heat conduction and diffusion equations with partially known eigenstructure

Jun Imai
Okayama University

Yasuaki Ando
Okayama University

Masami Konishi
Okayama University

Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

An optimal finite-diemensional modeling in heat conduction and diffusion equations with partially known eigenstructure

Jun Imai, Yasuaki Ando and Masami Konishi
Department of Electrical and Electronic Engineering, Okayama University
3-1-1 Tsushima-naka, Okayama, 700-8530 JAPAN
{imai,yasuaki,konishi}@cntr.elec.okayama-u.ac.jp

***Abstract*— An optimal finite-dimensional modeling technique is presented for a standard class of distributed parameter systems for heat and diffusion equations. A finite-dimensional nominal model with minimum error bounds in frequency domain is established for spectral systems with partially known eigenvalues and eigenfunctions. The result is derived from a completely characterized geometric figure upon complex plane, of all the frequency responses of the systems that have (i) a finite number of given time constants T_i's and modal coefficients k_i's, (ii) an upper bound ρ to the infinite sum of the absolute values of all the modal coefficients k_i's, (iii) an upper bound T to the unknown T_i's, and (iv) a given dc gain $G(0)$. Discussions are made on how each parameter mentioned above makes contribution to bounding error or uncertainty, and we stress that steady state analysis for dc input is used effectively in reduced order modeling and bounding errors. The feasibility of the presented scheme is demonstrated by a simple example of heat conduction in ideal copper rod.**

I. INTRODUCTION

For controller design synthesis in view of robust control theory, it is necessary to specify a nominal model describing essential plant dynamics and also bounds of magnitudes of the uncertainty [1]. A lot of efforts have been made to establish methodology of reduced order modeling and error bounding of spatially distributed systems [2]. A distributed parameter system described by a partial differential equation is of infinite-dimensional, and, in fact, an appropriate sort of finite-dimensional approximation is essential to achieve effective controller design synthesis. In most practical cases, just partial knowledge or incomplete data for a plant is available or can be made use of. How to obtain an effective model with error bound in such a situation is one of the fundamental issues on control of spatially distributed plant. An infinite number of parameters in the distributed parameter systems are hardly exactly known ; often at best, a few of relatively accurate parameters can be evaluated.

Erickson et al. [3] proposed an error bounding scheme in such a situation and developed a technique for modal truncation of spectral systems including parabolic and hyperbolic distributed parameter systems. Their results have been extended in [4] using dc gain, and new error bounds for a nominal model with feedthrough term have been developed. In view of relationship between error bound and information about the plant, a notion of feasible set of systems has been introduced for a class of hyperbolic distributed parameter systems, and least upper bounds of errors are established by evaluating the norm of the error as the size of a ball covering the set [5]. A feasible set of frequency responses has been explicitly characterized as a geometric figure in complex plane [6], trying to keep away from possible conservatisms caused by overestimating uncertainty.

For heat conduction and diffusion systems, dc gain information seems to have much more advantages rather than for flexible vibrating systems. Many researchers have attacked to improve the precision of reduced order models using such dc gain information thus far [7], but few discussions have ever been focused on the effectiveness of using dc gain in high frequency range. In this paper, we will discuss, with the spirit in [6], how steady state analysis for dc input play important roles in modeling and error bounding.

This paper is organized as follows. In section 2, systems are formulated as an infinite series of transfer functions, and a feasible set of systems is defined by using certain limited number of conditions. In the case where dc gain is not used, the feasible frequency responses are proved to be geometrically characterized by two arcs. Section 3 describes that the set for dc gain case is depicted as the convex hull of four arcs. A feature of the results is that it provides us the nominal model with the least additive error bound, and both of them are explicitly described by simple real rational form of transfer functions. In section 4, error bounds for three candidate nominal models are compared. Further bounds using another condition are discussed to diminish them to zero as frequency goes to infinity. For a spatially one-dimensional heat equation, we show in section 5 the way how the parameters necessary for error bounding are obtained using numerical integration. The feasibility of the presented scheme is demonstrated by a simple example of heat conduction of ideal copper rod.

Notation: By $\mathrm{ch}\,(A)$ we denote a convex hull of a set A on a complex plane, that is, a minimum convex set which contains A. For a vector $b \in \mathbf{C}$ and a set $A \subset \mathbf{C}$, the set $\{c \in \mathbf{C} | c = a + b, a \in A\}$ is just denoted by $A + b$.

II. SYSTEM FORMULATION AND PRELIMINARIES

A. A feasible set of systems

It is well known that a transfer function of a linear time-invariant system corresponding to heat conduction and diffusion, can be written as an infinite series of first order lag modes:

$$G(s) = \sum_{i=1}^{\infty} \frac{k_i}{1+T_i s} \tag{1}$$

where $T_i > 0$ and $T_1 \geq T_2 \geq \cdots \to 0$ and

$$\sum_{i=1}^{\infty} |k_i| \leq \rho \tag{2}$$

for some given $\rho > 0$. Here T_i is a time constant and k_i is a modal coefficient for each i-th mode. We assume that the first ℓ pairs of (k_i, T_i) for $i = 1, \ldots, \ell$, are given but all the rest (for $i = \ell + 1, \ldots,$) unknown.

Furthermore, let us assume it is verified that

$$0 < T_i \leq T \quad \text{for} \quad i > \ell \tag{3}$$

for some given $T \leq T_\ell$.

Denote the ℓ-th partial sum of $G(s)$, the known part, by

$$G_\ell(s) := \sum_{i=1}^{\ell} \frac{k_i}{1+T_i s} \tag{4}$$

and by $\mathcal{P}_\ell^{\rho,T}$ the set of all the systems written as equation (1) that have (i) a finite number of given time constants T_i's and modal coefficients k_i's, (ii) an upper bound ρ to the infinite sum of the absolute values of all the modal coefficients k_i's as in (2), and (iii) an upper bound T to the unknown T_i's as in (3) ; that is,

$$\mathcal{P}_\ell^{\rho,T} := \left\{ G(s) = \sum_{i=1}^{\infty} \frac{k_i}{1+T_i s} \middle| (k_i, T_i) \text{ fixed for } i \leq \ell \right.$$
$$\left. \sum_{j=1}^{\infty} |k_j| \leq \rho;\ 0 < T_i \leq T \text{ for } i > \ell \right\} \tag{5}$$

which we call a feasible set. We denote the set of all the frequency responses corresponding to the elements in the feasible set $\mathcal{P}_\ell^{\rho,T}$ by $\mathcal{P}_\ell^{\rho,T}(j\omega)$.

B. Feasible frequency responses

In this subsection, we will characterize $\mathcal{P}_\ell^{\rho,T}(j\omega)$, the set of all the possible frequency responses $G(j\omega)$ as a geometric figure on the complex plane, at any specified frequency ω.

Since

$$\mathcal{P}_\ell^{\rho,T} = \left\{ G(s) = G_\ell(s) + \tilde{G}(s) | \tilde{G}(s) \in \mathcal{P}_0^{\overline{\rho}^{(\ell)},T} \right\} \tag{6}$$

where

$$\overline{\rho}^{(\ell)} := \rho - \sum_{j=1}^{\ell} |k_j|, \tag{7}$$

the set $\mathcal{P}_\ell^{\rho,T}(j\omega)$ is a parallel translation of $\mathcal{P}_0^{\overline{\rho}^{(\ell)},T}(j\omega)$ by $G_\ell(j\omega)$, that is,

$$\mathcal{P}_\ell^{\rho,T}(j\omega) = \mathcal{P}_0^{\overline{\rho}^{(\ell)},T}(j\omega) + G_\ell(j\omega).$$

So it is enough to investigate $\mathcal{P}_0^{r,T}(j\omega)$.

Theorem 1. For each frequcnecy ω, the set $\mathcal{P}_0^{r,T}(j\omega)$ is characterized as follows:

$$\mathcal{P}_0^{r,T}(j\omega) = \text{ch}\left[\{\pm r H_\theta(j\omega) | 0 < \theta \leq T\}\right] \tag{8}$$

where

$$H_\theta(s) = \frac{1}{1+\theta s}. \tag{9}$$

This implies that the set $\mathcal{P}_0^{\overline{\rho}^{(\ell)},T}(j\omega)$ is depicted on the complex plane as the convex hull of the two arcs $A_1 := \{\overline{\rho}^{(\ell)} H_\theta(j\omega), 0 < \theta \leq T\}$ and $A_2 := \{-\overline{\rho}^{(\ell)} H_\theta(j\omega), 0 < \theta \leq T\}$.

In other words, if the bound ρ to the infinite series of the absolute of k_i and an upper bound T to unknown time constants $T_i' s$ are given, then the feasible set is depicted by just using two arcs.

Based on the above results, the next corollary is immediate.

Corollary 1. For each frequency ω, a complex number $G_{\rm n}(j\omega)$ that minimizes

$$\sup_{G \in \mathcal{P}_\ell^{\rho,T}} |G(j\omega) - G_{\rm n}(\omega)| \tag{10}$$

is no other than $G_\ell(j\omega)$, and

$$\min_{G_{\rm n}(\omega) \in \mathbf{C}} \sup_{G \in \mathcal{P}_\ell^{\rho,T}} |G(j\omega) - G_{\rm n}(\omega)| = \overline{\rho}^{(\ell)}. \tag{11}$$

We can see that the least upper error bound under the information of ρ, T, and up to ℓ-th eigenstructures, is proved to be a constant $\overline{\rho}^{(\ell)}$ that is independent of frequency ω.

III. Main results

A. Dc gain information

In addition to condition (i)-(iii) in the previous section, we consider hereafter the case where also (iv) the dc gain

$$d := G(0) = \sum_{i=1}^{\infty} k_i \tag{12}$$

is given. We are to establish how this dc gain information shrinks the feasible set.

The feasible set corresponding to this case is denoted by $\mathcal{P}_\ell^{\rho,T,d} := \mathcal{P}_\ell^{\rho,T} \cap \mathcal{D}_d$ where $\mathcal{D}_d := \{G(s) | G(0) = d\}$. Furthermore we define

$$\overline{d}^{(\ell)} := d - \sum_{i=1}^{\ell} k_i = G(0) - G_\ell(0), \tag{13}$$

and we can see it is enough to characterize $\mathcal{P}_0^{r,v,\delta}(j\omega)$ for given r, v, and δ since $\mathcal{P}_\ell^{\rho,T,d}(j\omega) = \mathcal{P}_0^{\overline{\rho}^{(\ell)},T,\overline{d}^{(\ell)}}(j\omega) + G_\ell(j\omega)$ similarly as in the previous section.

The next theorem is the main result of the paper and it provides a geometric characterization of possible frequency responses $\mathcal{P}_0^{r,v,\delta}(j\omega)$ on complex plane for any user-specified frequency.

Theorem 2. For any frequency ω, $\mathcal{P}_0^{\rho,T,d}(j\omega)$ coincide with the convex hull of the union of the following four arcs of circle segments

$$A_{1a}: \quad \left\{ \frac{d-\rho}{2} H_T(j\omega) + \frac{d+\rho}{2} H_\theta(j\omega) \middle| 0 < \theta \le T \right\}$$

$$A_{1b}: \quad \left\{ \frac{d-\rho}{2} H_\theta(j\omega) + \frac{d+\rho}{2} \middle| 0 < \theta \le T \right\}$$

$$A_{2a}: \quad \left\{ \frac{d+\rho}{2} H_T(j\omega) + \frac{d-\rho}{2} H_\theta(j\omega) \middle| 0 < \theta \le T \right\}$$

$$A_{2b}: \quad \left\{ \frac{d+\rho}{2} H_\theta(j\omega) + \frac{d-\rho}{2} \middle| 0 < \theta \le T \right\},$$

that is, $\mathcal{P}_0^{\rho,T,d}(j\omega) = \mathrm{ch}\,[A_{1a} \cap A_{1b} \cap A_{2a} \cap A_{2b}]$.

Proof See the next subsection.

From Theorem 2, by considering two most distant points in $\mathcal{P}_0^{\rho,T,d}(j\omega)$, we can easily see that a disk with minimum radius containing $\mathcal{P}_0^{\rho,T,d}(j\omega)$ is characterized by

$$\text{the center: } \frac{d}{2}\left(1 + \frac{1}{1+j\omega T}\right), \text{ and}$$

$$\text{the radius: } \frac{\rho}{2}\left(\frac{\omega T}{\sqrt{1+(\omega T)^2}}\right).$$

The circle of the disk boundary is tangent to $\mathcal{P}_0^{\rho,T,d}(j\omega)$, and the tangent points are

$$T_+(j\omega) := G_\ell(j\omega) + \frac{\overline{d}^{(\ell)} + \overline{\rho}^{(\ell)}}{2} H_T(j\omega) + \frac{\overline{d}^{(\ell)} - \overline{\rho}^{(\ell)}}{2}$$

$$T_-(j\omega) := G_\ell(j\omega) + \frac{\overline{d}^{(\ell)} - \overline{\rho}^{(\ell)}}{2} H_T(j\omega) + \frac{\overline{d}^{(\ell)} + \overline{\rho}^{(\ell)}}{2}.$$

Then we can easily see a result for dc gain case corresponding to Theorem 1 is stated as follows:

Corollary 2. For each frequency ω, a complex number $G_n(\omega)$ that minimizes $\sup_{G \in \mathcal{P}_\ell^{\rho,T,d}} |G(j\omega) - G_n(\omega)|$ is

$$G_\ell(j\omega) + \frac{\overline{d}^{(\ell)}}{2}\left(1 + \frac{1}{1+j\omega T}\right) \tag{14}$$

and the minimum is given by

$$\min_{G_n(\omega)\in \mathbb{C}} \sup_{G \in \mathcal{P}_\ell^{\rho,T,d}} |G(j\omega) - G_n(\omega)| = \frac{\overline{\rho}^{(\ell)}}{2}\left(\frac{\omega T}{\sqrt{1+(\omega T)^2}}\right). \tag{15}$$

B. A sketch of proof of Theorem 2

The shape of the feasible frequency responses for fixed time constants T_i's becomes a polygon. It is enough to identify the extreme points of possible polygons to characterize the feasible set.

We can show the problem is reduced to the inclusion by two terms, as follows.

Lemma 1. The following relation holds true.

$$\mathcal{P}_0^{\rho,T,d}(j\omega) = \mathrm{ch}\,[\mathcal{S}_0^{\rho,T,d}(j\omega)] \tag{16}$$

where

$$\mathcal{S}_0^{\rho,T,d}(j\omega) := \{k_a H_{T_a}(j\omega) + k_b H_{T_b}(j\omega) | \; 0 < T_b \le T_a \le T, k_a + k_b = d, |k_a| + |k_b| = \rho\} \tag{17}$$

$\mathcal{S}_0^{\rho,T,d}(j\omega)$ is a set of candidate extreme points for $\mathcal{P}_0^{\rho,T,d}(j\omega)$.

Our approach is to enumerate all the candidate extreme points that are on the interior or boundary of the set, and then the convex hull of the extreme points coincides with the feasible set.

On the other hand, to evaluate the convex hull, we see that every candidate extreme points move along circles if time constants are moved in the interval. The convex hull in (8) is characterized by circle segments as shown next

Lemma 2. $\mathcal{S}_0^{\rho,T,d}(j\omega)$ is met the following relations.

$$\mathcal{S}_0^{\rho,T,d}(j\omega) = \bigcup_{0<\theta\le T} \underline{\mathcal{S}_0^{\rho,\theta,d}}(j\omega) \tag{18}$$

where

$$\underline{\mathcal{S}_0^{\rho,\theta,d}}(j\omega) := \{k_a H_\theta(j\omega) + k_b H_{T_b}(j\omega) | \; 0 < T_b \le \theta, k_a + k_b = d, |k_a| + |k_b| = \rho\} \tag{19}$$

Lemma 3. $\underline{\mathcal{S}_0^{\rho,\theta,d}}(j\omega)$ is given by the convex hull of the union of the following two arcs:

$$A_a^{(\theta)}: \left\{ \frac{d-\rho}{2} H_\theta(j\omega) + \frac{d+\rho}{2} H_\sigma(j\omega) \middle| \sigma \ge \theta \right\}$$

$$A_b^{(\theta)}: \left\{ \frac{d-\rho}{2} H_\sigma(j\omega) + \frac{d+\rho}{2} H_\theta(j\omega) \middle| \sigma \ge \theta \right\}$$

Proof is straightforward using above preparation.

IV. Application to Model Selection and Error Bounding

A. Additive uncertainty model

Corollary 2 in the previous section implies that the additive uncertainty model

$$\{G(s) = G_n(s) + W(s)\Delta(s) | \; \Delta(s) \in RH^\infty, \|\Delta\|_\infty \le 1\} \tag{20}$$

with the nominal

$$G_n(s) = G_\ell(s) + \frac{G(0) - G_\ell(0)}{2}\left(1 + \frac{1}{1+Ts}\right) \tag{21}$$

and uncertainty weight

$$W(s) = \frac{\overline{\rho}^{(\ell)}}{2}\left(\frac{Ts}{1+Ts}\right) \tag{22}$$

has the minimum magnitudes of weight $|W(j\omega)|$ for every frequency ω, over a class of all the additive uncertainty models containing the feasible set $\mathcal{P}_\ell^{\rho,T,d}(j\omega)$.

A candidate for finite-dimensional approximating model with additive uncertainty under the conditions (i)–(iv), is immediate from Corollary 2. Other candidates and their bounding results are presented in the next subsection, and based on the result we can select an uncertainty model and then proceed to design a controller by linear robust control theory.

B. Nominal models and error bounds

We have to often take into account the order and relative degree for choosing a nominal model. Since our result here characterizes the feasible set in frequency domain, the least upper bounds for the error between the system and any specified nominal model is readily computable at any user-specified frequencies.

For a nominal model $G_\ell(s)+G(0)-G_\ell(0)$, we have had the following error bound [4]:

$$|G(j\omega)-(G_\ell(j\omega)+G(0)-G_\ell(0))| \leq \overline{\rho}^{(\ell)}\cdot\frac{\omega T}{\sqrt{1+(\omega T)^2}} \tag{23}$$

But, in fact, by utilizing the geometric shape from Theorem 2 we have immediately the following:

Corollary 3. For each frequency ω,

$$\sup_{G\in\mathcal{P}_\ell^{\rho,T,d}} |G(j\omega)-(G_\ell(j\omega)+G(0)-G_\ell(0))| = \frac{\overline{\rho}^{(\ell)}+|G(0)-G_\ell(0)|}{2}\cdot\frac{\omega T}{\sqrt{1+(\omega T)^2}} \tag{24}$$

It is often required for a design model that, for example, the transfer function have no feedthrough term and therefore be strictly proper. Choosing a nominal model that meets such a condition yields a bounding result as in the following:

Corollary 4. Let us take a nominal model as

$$G_n(s)=G_\ell(s)+\frac{G(0)-G_\ell(0)}{1+Ts}. \tag{25}$$

Then for each frequency ω,

$$\sup_{G\in\mathcal{P}_\ell^{\rho,T,d}} |G(j\omega)-G_n(\omega)| = \left(\frac{|G(0)-G_\ell(0)|+\overline{\rho}^{(\ell)}}{2}\right)\cdot\left(\frac{\omega T}{\sqrt{1+(\omega T)^2}}\right) \tag{26}$$

A user may feel intuitively that something goes wrong because it is not along a postulate that the error should converge to zero as frequency tends to large. We can easily see that it is not possible unless other information is introduced. Suppose we consider an additional condition (v); for a given $\sigma>0$

$$\sum_{i=1}^{\infty}|k_i/T_i| \leq \sigma \tag{27}$$

is verified[1].

[1]For evaluating such a σ, see the later sections.

Then we define another feasible set $\mathcal{Q}_\ell^{\sigma,T}$ as follows:

$$\mathcal{Q}_\ell^{\sigma,T} := \left\{ G(s)=\sum_{i=1}^{\infty}\frac{k_i}{1+T_is} \middle| (k_i,T_i) \text{ fixed for } i\leq\ell \right. \\ \left. \sum_{j=1}^{\infty}|k_j/T_j| \leq \sigma; 0<T_j\leq T\ (j>\ell) \right\} \tag{28}$$

Theorem 3. For each frequency ω, the feasible frequency responses for $\mathcal{Q}_0^{\overline{\sigma}^{(\ell)},T}$ is represented as

$$\mathcal{Q}_0^{\overline{\sigma}^{(\ell)},T}(j\omega) = \mathrm{ch}\left[\{(k/\theta)H_\theta(j\omega)\,|\,|k|\leq\overline{\sigma}^{(\ell)}, 0<\theta\leq T\}\right] \tag{29}$$

where

$$\overline{\sigma}^{(\ell)} := \sigma-\sum_{j=1}^{\ell}|k_j/T_j|. \tag{30}$$

That is, $\mathcal{Q}_0^{\overline{\sigma}^{(\ell)},T}(j\omega)$ is depicted by two arcs (1) $\{(\overline{\sigma}^{(\ell)}\theta)H_\theta(j\omega), 0<\theta\leq T\}$ and (2) $\{-(\overline{\sigma}^{(\ell)}\theta)H_\theta(j\omega), 0<\theta\leq T\}$ and their convex hull.

Corollary 5. Let us take

$$G_n(s)=G_\ell(s)+\frac{G(0)-G_\ell(0)}{1+Ts}.$$

Then

$$\sup_{G\in\mathcal{Q}_\ell^{\sigma,T}} |G(j\omega)-G_n(\omega)| \tag{31}$$

is given by

$$\left(|G(0)-G_\ell(0)|+\frac{\overline{\sigma}^{(\ell)}}{T}\right)\cdot\left(\frac{1}{\sqrt{1+(\omega T)^2}}\right). \tag{32}$$

V. Reduced Order Modeling of Parabolic Distributed Parameter Systems

A. One-dimensional heat conduction problem

It may be beneficial to grasp some ideas by seeing a simple example as follows. Let us consider a temperature distribution $v(t,\xi)$ at time $t>0$ on one-dimensional spatial coordinate $\xi\in[0,1]$, which is governed by a partial differential equation

$$\frac{\partial v}{\partial t}(t,\xi)=\frac{\partial}{\partial\xi}\left(f(\xi)\frac{\partial v}{\partial\xi}(t,\xi)\right)-\beta(\xi)v(t,\xi)+b(\xi)u(t) \tag{33}$$

with boundary conditions

$$v(t,1)=v(t,0)=0 \tag{34}$$

and measurement output

$$y(t)=\int_0^1 c(\xi)v(t,\xi)d\xi \tag{35}$$

where $f(\xi)$ and $\beta(\xi)$ are smooth nonnegative functions, and $b(\xi)$ and $c(\xi)$ are smooth non-negative functions that describe spatial heating and temperature measurement effect, respectively. Here $u(t)$ represents the heater input power and $y(t)$ the sensor output at time t.

A self-adjoint closed operator $\mathcal{A}$ is defined on $L^2(0,1)$ as

$$\mathcal{A}v = \frac{\partial}{\partial \xi}\left(f(\xi)\frac{\partial v}{\partial \xi}(t,\xi)\right) - \beta(\xi)v(t,\xi) \quad (36)$$

$$v \in D(\mathcal{A}) = \{v \in L^2(0,1) : \frac{\partial^2 v}{\partial \xi^2} \in L^2(0,1), \frac{\partial v}{\partial \xi}(1) = \frac{\partial v}{\partial \xi}(0) = 0\} \quad (37)$$

and an eigenvalue λ_i and a corresponding eigenfunction ϕ_i are defined by the relation

$$\mathcal{A}\phi_i = \lambda_i \phi_i \quad (38)$$

where $|\lambda_1| \leq |\lambda_2| \leq \cdots$. Then we define $\mathcal{B} : \mathbf{R} \to L^2(0,1)$, and $\mathcal{C} : L^2(0,1) \to \mathbf{R}$ as

$$\mathcal{B}u(t) = b(\xi)u(t), \quad \mathcal{C}v = \int_0^1 c(\xi)v(\xi)d\xi \quad (39)$$

and an adjoint operator $\mathcal{B}^* : L^2(0,1) \to \mathbf{R}$ of $\mathcal{B}$ is represented as

$$\mathcal{B}^*v = \int_0^1 b(\xi)v(\xi)d\xi. \quad (40)$$

If we define $b_i := \mathcal{B}^*\phi_i$, $c_i := \mathcal{C}\phi_i$, then the transfer function of the system from u to y is written as follows:

$$G(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} = \sum_{i=1}^{\infty} \frac{c_i b_i}{s - \lambda_i} \quad (41)$$

Note that the system is of infinite-dimensional. For spatially varying case it requires numerical analysis such as finite element methods to evaluate c_i, b_i, and λ_i. Generally, highly accurate values are hardly computable for higher order i. It is well-known that for a self-adjoint operator $\mathcal{A}$, eigenparameter bounds are numerically available utilizing variational principles [8,9].

In such a situation, our task is to develop a modeling technique to yield a finite-dimensional approximating model and also error bounds that serve efficient controller design.

B. Example

We consider an example for modeling of heat conduction of a copper rod. Dynamics of temperature distribution $v(t,\xi)$[K] is described as

$$v_t(t,\xi) = \alpha^2 v_{\xi\xi}(t,\xi) - \beta v(t,\xi) + b(\xi)u(t) \quad (0 < \xi < 1), \quad (42.a)$$

$$v(t,0) = v(t,1) = 0, \quad (42.b)$$

$$y(t) = \int_0^1 c(\xi)v(t,\xi)d\xi. \quad (42.c)$$

Furthermore

$$b(\xi) = \begin{cases} \eta & \text{for } \xi \in [0,l] \\ 0 & \text{otherwise,} \end{cases}$$

$$c(\xi) = \begin{cases} 1/\epsilon & \text{for } |\xi - p| \leq \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha^2 = 1.16 \times 10^{-4}$, $\beta = 0.018$, $\eta = 0.0140$, $l = 1/5$m, $p = 0.3$m, and $\epsilon = 1/500$[1/m].

Table 1 Modal parameters and intermediate computed upper bounds to $\sum_i |k_i/T_i|$ (see appendix) .

i	k_i	T_i
1	0.0497	319.9
2	0.0900	272.2
3	0.0296	0.116
4	-0.0457	4.087
5	-0.0537	0.591
6	-0.0188	0.614
7	0.00490	0.614

γ	δ	upper bound
1	0	0.140
1/2	0	0.714
1/2	1/2	0.116
0	0	4.087
0	1/2	0.591
0	1	0.614

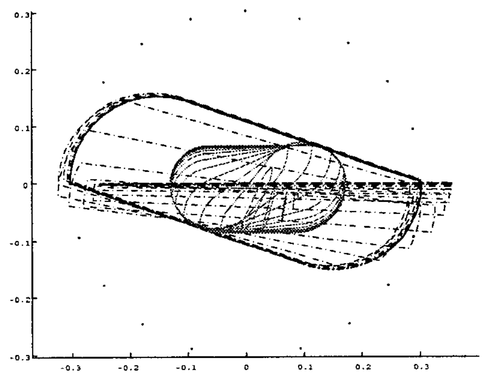

Fig. 1. The feasible sets (solid) with and without (dashed) dc information ($\omega : 10^{-4} - 10^{4}$ rad/sec).

We suppose $\ell = 6$ and just six eigenstructures are known: $T_1 = 319.9$s, $T = T_2 = 272.2$s, $k_1 = 0.04974$, $d = 0.0945$, $\rho = 0.591$, $\overline{\rho}^{(6)} = 0.541$, $\overline{d}^{(6)} = 4.448 \times 10^{-2}$. The shape of feasible set at frequency $\omega = 1.0 \times 10^{-4}$ rad/sec is depicted in Figure 2. The frequency characteristic of the radii of the feasible set with and without dc gain are compared in Figure 3. Error bounds of the proposed nominal model in Corollary 1 through 5 are also plotted in the same figure.

VI. Conclusion

In this paper we presented a reduced order modeling of uncertainty for heat and diffusion equations. Here we presented a method to characterize uncertainty as a feasible set in the frequency domain. We showed that the shape of the bounded set of all the complex numbers of frequency responses of the systems that satisfy the condition is depicted by several circle segments.

Theoretical limitation was clarified about the minimum additive uncertainty of any nominal models under the information given. The set theoretic characterization enables us to develop new results that the information of dc gain of the system will effectively shrink the size of the feasible set.

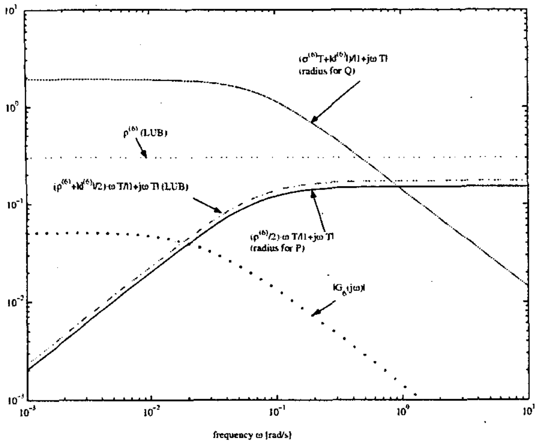

Fig. 2. Comparison between sizes of feasible sets and the bound

VII. REFERENCES

[1] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. Macmillan, 1992.

[2] A. Helmicki, C. Jacobson, and C. Nett, "Control oriented system identification: A worst-case/deterministic approach in h_∞," *IEEE Trans. Autom. Control*, vol. 36, no. 10, pp. 1163–1176, 1991.

[3] M. A. Erickson, R. S. Smith, and A. J. Laub, "Finite-dimensional approximation and error bounds for spectral systems with partially known eigenstructure," *IEEE Trans. Automatic Control*, vol. AC-39, no. 9, pp. 1904–1909, 1994.

[4] J. Imai and K. Wada, "A finite dimensional modeling with explicit error bounds for spectral systems," *35th IEEE Conf. on Decision and Control*, pp. 400–403, 1996.

[5] J. Imai and K. Wada, "Modeling uncertainty of large flexible structures with unknown modal parameters," *39th IEEE Conf. on Decision and Control*, pp. 3780–3784, 2000.

[6] J. Imai, "Modeling flexible structures with unknown high order modal parameters: a feasible set approach in frequency domain," *41st IEEE Conf. on Decision and Control*, pp. 4092–4094, 2002.

[7] G. Obinata and B. D. O. Anderson, *Model Reduction for Control System Design*. Springer-Verlag, 2000.

[8] S. H. Gould, *Variational Methods for Eigenvalue Problems: an introduction to the Methods of Rayleigh, Ritz, Weinstein, and Aronszajn*. University of Tronto Press, 1957.

[9] H. F. Weinberger, "Error ounds in the rayleigh-ritz approximation of eigenvectors," *J. Research Nat. Bureau Standards*, vol. 64B, no. 4, pp. 217–225, 1960.

APPENDIX

We mentioned $\rho > 0$ such that $\sum_i |k_i| \le \rho$ and $\sigma > 0$ such that $\sum_{i=1}^{\infty} |k_i/T_i| \le \sigma$. Let us see here to evaluate upper bounds to $\sum_i |k_i T_i^{-\gamma}|$ ($\gamma = 0, 1/2, 1$) just using $b, \bar{b}, c, \bar{c}$ where $-\mathcal{A}\bar{b} = b$,and $-\mathcal{A}\bar{c} = c$. Since for $0 \le \delta \le 1-\gamma$, ($\gamma, \delta \in [0,1]$)

$$\sum_i |k_i T_i^{-\gamma}| = \sum_i |c_i b_i||\lambda_i|^{1-\gamma} = \sum_i |c_i||\lambda_i|^{\delta}\cdot|b_i||\lambda_i|^{1-\gamma-\delta}$$

$$\le (\sum_i |c_i|^2|\lambda_i|^{2\delta})^{1/2}(\sum_i |b_i|^2|\lambda_i|^{2(1-\gamma-\delta)})^{1/2} =: \sigma_{\gamma,\delta}$$

according to the Hölder's inequality. On the other hand, The Parsival's equality says:

$$\sum_i |c_i|^2 = \int_0^1 c^2(\xi)d\xi, \quad \sum_i \left(\frac{|b_i|}{-\lambda_i}\right)^2 = \int_0^1 \bar{b}^2(\xi)d\xi.$$

The right hand sides are computationally tractable. A possible choice is to take

$$\rho = \min_{\delta}\{\sigma_{0,\delta}|\delta = 0, 1/2, 1\}$$

$$\sigma = \sigma_{1,0}$$

Upper bounds are derived as in Table A.

Note that $d = G(0)$ can be evaluated as follows:

Table A Upper bounds to $\sum_i |k_i T_i^{-\gamma}|$ ($\gamma = 0, 1/2, 1$).

γ	δ	upper bound
1	0	$\sqrt{\int c^2 d\xi \cdot \int b^2 d\xi}$
1/2	0	$\sqrt{\int c^2 d\xi \int \bar{b} b d\xi}$
1/2	1/2	$\sqrt{\int \bar{c} c d\xi \int b^2 d\xi}$
0	0	$\sqrt{\int c^2 d\xi \int \bar{b}^2 d\xi}$
0	1/2	$\sqrt{\int \bar{c} c d\xi \int \bar{b} b d\xi}$
0	1	$\sqrt{\int \bar{c}^2 d\xi \int b^2 d\xi}$