high order error analysis, *Int. J. Comp. Vis.* **80**-2 (2008-11), 167–188.

[15] K. Kanatani and N. Ohta: Automatic detection of circular objects by ellipse growing, *Int. J. Image Graphics*, **4**-1 (2004-1), 35–50.

[16] K. Kanatani and Y. Sugaya: Performance evaluation of iterative geometric fitting algorithms, *Comp. Stat. Data Anal.*, **52**-2 (2007-10), 1208–1222.

[17] K. Kanatani and Y. Sugaya: Compact algorithm for strictly ML ellipse fitting, *Proc. 19th Int. Conf. Pattern Recognition*, Tampa, FL, U.S.A., TuBCT8.52, December (2008).

[18] Y. Leedan and P. Meer: Heteroscedastic regression in computer vision: Problems with bilinear constraint, *Int. J. Comput. Vision.*, **37**-2 (2000-6), 127–150.

[19] B. C. Matei and P. Meer: Estimation of nonlinear errors-in-variables models for computer vision applications, *IEEE Trans. Patt. Anal. Mach. Intell.*, **28** (2006-10), 1537–1552.

[20] K. A. Paton, Conic sections in chromosome analysis: *Patt. Recog.*, **2**-1 (1970-1), 39–40.

[21] P. Rangarajan and K. Kanatani: Improved algebraic methods for circle fitting, *Elec. J. Stat.*, **3** (2009), 1075–1082.

[22] P. L. Rosin: A note on the least squares fitting of ellipses, *Patt. Recog. Lett.*, **14**-10 (1993-10), 799-808.

[23] P. L. Rosin and G. A. W. West: Nonparametric segmentation of curves into various representations, *IEEE. Trans. Patt. Anal. Mach. Intell.*, **17**-12 (1995-12), 1140-1153.

[24] G. Taubin: Estimation of planar curves, surfaces, and non-planar space curves defined by implicit equations with applications to edge and range image segmentation, *IEEE Trans. Patt. Anal. Mach. Intell.*, **13**-11 (1991-11), 1115–1138.

Appendix

The expectation $E[\Delta_1 M \bar{M}^- \Delta_1 M]$ is computed as follows:

$$E[\Delta_1 M \bar{M}^- \Delta_1 M]$$

$$= E[\frac{1}{N}\sum_{\alpha=1}^{N}\left(\bar{\xi}_\alpha \Delta_1 \xi_\alpha^\top + \Delta_1 \xi_\alpha \bar{\xi}_\alpha^\top\right)\bar{M}^- \frac{1}{N}\sum_{\beta=1}^{N}\left(\bar{\xi}_\beta \Delta_1 \xi_\beta^\top\right.$$
$$\left.+ \Delta_1 \xi_\beta \bar{\xi}_\beta^\top\right)]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} E[(\bar{\xi}_\alpha \Delta_1 \xi_\alpha^\top + \Delta_1 \xi_\alpha \bar{\xi}_\alpha^\top)\bar{M}^-(\bar{\xi}_\beta \Delta_1 \xi_\beta^\top$$
$$+ \Delta_1 \xi_\beta \bar{\xi}_\beta^\top)]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} E[\bar{\xi}_\alpha \Delta_1 \xi_\alpha^\top \bar{M}^- \bar{\xi}_\beta \Delta_1 \xi_\beta^\top + \bar{\xi}_\alpha \Delta_1 \xi_\alpha^\top \bar{M}^- \Delta_1 \xi_\beta \bar{\xi}_\beta^\top$$
$$+ \Delta_1 \xi_\alpha \bar{\xi}_\alpha^\top \bar{M}^- \bar{\xi}_\beta \Delta_1 \xi_\beta^\top + \Delta_1 \xi_\alpha \bar{\xi}_\alpha^\top \bar{M}^- \Delta_1 \xi_\beta \bar{\xi}_\beta^\top]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} E[\bar{\xi}_\alpha(\Delta_1 \xi_\alpha, \bar{M}^- \bar{\xi}_\beta)\Delta_1 \xi_\beta^\top$$
$$+ \bar{\xi}_\alpha(\Delta_1 \xi_\alpha, \bar{M}^- \Delta_1 \xi_\beta)\bar{\xi}_\beta^\top + \Delta_1 \xi_\alpha(\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\beta)\Delta_1 \xi_\beta^\top$$
$$+ \Delta_1 \xi_\alpha(\bar{\xi}_\alpha, \bar{M}^- \Delta_1 \xi_\beta)\bar{\xi}_\beta^\top]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} E[(\Delta_1 \xi_\alpha, \bar{M}^- \bar{\xi}_\beta)\bar{\xi}_\alpha \Delta_1 \xi_\beta^\top$$
$$+ (\Delta_1 \xi_\alpha, \bar{M}^- \Delta_1 \xi_\beta)\bar{\xi}_\alpha \bar{\xi}_\beta^\top + (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\beta)\Delta_1 \xi_\alpha \Delta_1 \xi_\beta^\top$$
$$+ \Delta_1 \xi_\alpha(\bar{M}^- \Delta_1 \xi_\beta, \bar{\xi}_\alpha)\bar{\xi}_\beta^\top]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} E[\bar{\xi}_\alpha((\bar{M}^- \bar{\xi}_\beta)^\top \Delta_1 \xi_\alpha)\Delta_1 \xi_\beta^\top$$
$$+ \text{tr}[\bar{M}^- \Delta_1 \xi_\beta \Delta_1 \xi_\alpha^\top]\bar{\xi}_\alpha \bar{\xi}_\beta^\top + (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\beta)\Delta_1 \xi_\alpha \Delta_1 \xi_\beta^\top$$
$$+ \Delta_1 \xi_\alpha(\Delta_1 \xi_\beta^\top \bar{M}^- \bar{\xi}_\alpha)\bar{\xi}_\beta^\top]$$

$$= \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}\left(\bar{\xi}_\alpha \bar{\xi}_\beta^\top \bar{M}^- E[\Delta_1 \xi_\alpha \Delta_1 \xi_\beta^\top]\right.$$
$$+ \text{tr}[\bar{M}^- E[\Delta_1 \xi_\beta \Delta_1 \xi_\alpha^\top]]\bar{\xi}_\alpha \bar{\xi}_\beta^\top$$
$$+ (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\beta)E[\Delta_1 \xi_\alpha \Delta_1 \xi_\beta^\top]$$
$$\left.+ E[\Delta_1 \xi_\alpha \Delta_1 \xi_\beta^\top]\bar{M}^- \bar{\xi}_\alpha \bar{\xi}_\beta^\top\right)$$

$$= \frac{\sigma^2}{N^2}\sum_{\alpha,\beta=1}^{N}\left(\bar{\xi}_\alpha \bar{\xi}_\beta^\top \bar{M}^- \delta_{\alpha\beta}V_0[\xi_\alpha]\right.$$
$$+ \text{tr}[\bar{M}^- \delta_{\alpha\beta}V_0[\xi_\alpha]]\bar{\xi}_\alpha \bar{\xi}_\beta^\top + (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\beta)\delta_{\alpha\beta}V_0[\xi_\alpha]$$
$$\left.+ \delta_{\alpha\beta}V_0[\xi_\alpha]\bar{M}^- \bar{\xi}_\alpha \bar{\xi}_\beta^\top\right)$$

$$= \frac{\sigma^2}{N^2}\sum_{\alpha=1}^{N}\left(\bar{\xi}_\alpha \bar{\xi}_\alpha^\top \bar{M}^- V_0[\xi_\alpha]\right.$$
$$+ \text{tr}[\bar{M}^- V_0[\xi_\alpha]]\bar{\xi}_\alpha \bar{\xi}_\alpha^\top + (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\alpha)V_0[\xi_\alpha]$$
$$\left.+ V_0[\xi_\alpha]\bar{M}^- \bar{\xi}_\alpha \bar{\xi}_\alpha^\top\right)$$

$$= \frac{\sigma^2}{N^2}\sum_{\alpha=1}^{N}\left(\text{tr}[\bar{M}^- V_0[\xi_\alpha]]\bar{\xi}_\alpha \bar{\xi}_\alpha^\top + (\bar{\xi}_\alpha, \bar{M}^- \bar{\xi}_\alpha)V_0[\xi_\alpha]\right.$$
$$\left.+ 2\mathcal{S}[V_0[\xi_\alpha]\bar{M}^- \bar{\xi}_\alpha \bar{\xi}_\alpha^\top]\right) \tag{57}$$