Convergence Condition of Explicit Finite Element Method

for Heat Transfer Problem

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SYNOPSIS

The convergence condition of the explicit difference method for the heat transfer problem is already obtained. On the other hand, if the problem is formulated by using the weighted residual method for spatial axis, we have no tool to estimate the critical time-step width. In this paper, the estimation method is theoretically presented, and its propriety is examined through a number of numerical experiments.

1. INTRODUCTION

Heat Transfer Equation is the one which governs the heat transfer phenomenon, and in the field of Civil Engineering, many phenomena are governed by the same equation, for example, the seepage problem, the problem of the salt wedge in the ground water, the water pollution and the thermal stress in massive concrete. And, their solutions are desired.

The development of the digital computer in recent years has caused that of the numerical analysis, and, at present most of governing equations which present actual engineering problems can be approximately solved. These approximate solutions are mainly obtained by use of the Finite Difference Method (FDM) and the Weighted Residual Method (WRM) including the finite element method (FEM). Both of them express the solution of the governing equation at approximate nodal values located in space and time domain. Historically, the former is

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firstly developed, and it is widely applied in many engineering field. But, at its application to actual problem we find many difficulties for locating node in space, for example the settlement of grid nodes, the treatment of the complicated boundary configuration, and the introduction of the boundary conditions and so on. The Weighted Residual Method which is lately developed is advantageous comparing to FDM, because most of the complexities encountered at the application of FDM disappear in WRM. Therefore, in recent years the discretization of spatial axes is mainly done by use of WRM. On the other hand, the discretization along the time axis for most of governing equation is done by applying the FDM. The reason is that FDM can prepare a number of effective procedures and suitable one among them can be arbitrarily selected. Especially, for rather large scale problems the forward difference scheme is preferably used, because the objective linear equations need not to be solved but variables are explicitly determined. In the case of using the explicit method along the time axis and also any finite difference scheme along the spatial axes, the conditional equation on the stability of the numerical solution has been made clear. But in case that FEM is applied for the discretization of spatial axes and the forward difference scheme is used for that of the time axis, the stability condition of the numerical solution is not yet clarified.

In this paper, firstly the heat transfer equation is formulated by using the Galerkin method along the spatial axes and by the forward-difference scheme along time axis. Successively, the stability condition of the numerical solution for this mixed type of equations is theoretically introduced by using "Brauer's Theorem", and we discuss the merits of using the mixed-type of formulation (for the heat transfer problem) comparing to the stability condition according to FDM. The results are examined through a number of numerical experiments, and we show that the mixed type formulation can not only make the discretization of spatial and time axes ease but also elongate the width of the time-step comparing to the one by use of only FDM.

2. THEORETICAL ESTIMATION OF THE CRITICAL TIME STEP WIDTH

2.1 The Mixed Use of The Finite Element Method and The Finite Difference Method
Let's express the heat transfer equation as

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial t} \tag{2.1}
\]

with the boundary condition of eq. (2.2) which is illustrated in Fig. 2-2.

\[
\phi = \bar{\phi} \quad \text{on } S_1 \\
\frac{\partial \phi}{\partial n} = -h(\phi - v) \quad \text{on } S_2 \tag{2.2}
\]

where \( h \) is the heat transfer coefficient and \( v \) is the value of \( \phi \) out of the region.

![Fig. 2-1 Boundary condition](image)

Firstly we try discretize eq.'s (2.1) and (2.2) by using WRM along the spatial axes and FDM along the time axis, respectively. At the application of WRM we fix the time axis, that is, the right-hand of eq. (2.1) can be expressed as

\[
\frac{\partial \phi}{\partial t} = T \text{ (constant)} \tag{2.3}
\]

Then, we may treat eq. (2.4) instead of eq. (2.1).

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - T = 0 \tag{2.4}
\]

Application of The Galerkin Method on eq. (2.4) yields to

\[
\int\int (\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - T) \delta \phi \, dx \, dy = \int_{S_2} (\frac{\partial \phi}{\partial n} + h(\phi - v)) \delta \phi \, ds \tag{2.5}
\]
Integration of eq. (2.5) gives following expression;

\[
\iint \left\{ \frac{\partial \phi}{\partial x} \frac{\partial \delta \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \delta \phi}{\partial y} \right\} \, dx \, dy + \iint \Gamma \, dx \, dy = - \int_{s_2} h(\psi - \nu) \delta \phi \, ds
\]

(2.6)

Let's express above equation in matrix form. For this purpose we introduce eq. (2.7).

\[
\phi = [N]^T \{ \phi \}
\]

(2.7)

where \([N]\) is the shape function matrix and \(\{ \phi \}\) is the node value vector of an element.

Introduction of eq. (2.7) into eq. (2.6) gives the following matrix expression.

\[
\iint \left( \frac{\partial [N]}{\partial x} \frac{\partial [N]^T}{\partial x} + \frac{\partial [N]}{\partial y} \frac{\partial [N]^T}{\partial y} \right) \, dx \, dy [\phi] + \iint \Gamma [N] \, dx \, dy T
\]

\[
= - h \int_{s_2} [N][N]^T \, ds + h \int_{s_2} [N] \, ds \, v
\]

(2.8)

Now, we consider on the discretization along the time axis. Since the spatial axes and the time axis are discretized separately, we set \(T = \{ T \}\) and replace \(\{ T \}\) by \(\{ \phi_i \} - \{ \phi_{i-1} \}\) \(\Delta t\).

\[
\frac{\{ \phi_i \} - \{ \phi_{i-1} \}}{\Delta t}
\]

(2.9)

where \(\Delta t\) is the discretization width along the time axis and \(\{ \phi_i \}\), \(\{ \phi_{i-1} \}\) are the grid value vector at the \(i, i-1\) time steps respectively.

Eq. (2.9) is introduced in eq. (2.10) and we obtain

\[
\iint \left( \frac{\partial [N]}{\partial x} \frac{\partial [N]^T}{\partial x} + \frac{\partial [N]}{\partial y} \frac{\partial [N]^T}{\partial y} \right) \, dx \, dy [\phi_{i-1}]
\]

\[
+ \iint \frac{1}{\Delta t} [N] \, dx \, dy \{ \phi_i\} - \{ \phi_{i-1} \}
\]

\[
= - h \int_{s_2} [N][N]^T \, ds \{ \phi_{i-1} \} + h \int_{s_2} [N] \, ds \, v
\]

(2.10)
Finally, eq.(2.10) is expressed as the matrix equation;

\[
K\phi_{i-1} + C(\phi_i - \phi_{i-1}) = -B_1\phi_{i-1} + B_2
\]

where

\[
K = \int \left( \frac{\partial \mathbf{N}}{\partial x} \mathbf{N}^T + \frac{\partial \mathbf{N}}{\partial y} \mathbf{N}^T \right) \, dx \, dy
\]

\[
C = \frac{1}{\Delta t} \int \mathbf{N} \, dx \, dy
\]

\[
B_1 = h \int_{s_2} [\mathbf{N}] [\mathbf{N}]^T \, ds
\]

\[
B_2 = h \int_{s_2} [\mathbf{N}] \, ds 
\]

That is, eq.(2.11) is equivalent to following equation whose solution vector, \( \phi_i \), is directly obtained by introducing \( \phi_{i-1} \).

\[
\phi_i = A\phi_{i-1} + B
\]

where \( A = C^{-1}(C-K-B_1) \)

\[
B = B_2
\]

2.2 The Critical Time - Step Width

From eq.(2.14) we obtain following expression;

\[
\phi_i = A^{i} \phi_0 + (I-A)^{-1}(I-A^{i})B
\]

where \( \phi_i \) is the value of \( \phi \) at the \( i \)-th time step and \( \phi_0 \) is the initial value.

Eq.(2.15) clarifies that the convergence of \( \phi_i \) to a definite value, i.e. \( (I-A)^{-1}B \), is wholly governed by the property of the matrix \( A \). If the maximum eigenvalue, \( \lambda_{\text{max}} \), of \( A \) satisfies following relation, then above equation converges. (5)
Let's examine the maximum eigenvalue of the matrix \( A \) in eq. (2.13). From eq. (2.14)

\[
A = (I - C^{-1}K)
\]

where \( K = (K - B_1) \).

Then, the \( s \)-th row of \( A \) is explicitly expressed as following;

\[
[\Delta t K_s, 1, \Delta t K_s, 2, \ldots, \Delta t K_s, s - 1, 1 - \Delta t K_s, s, \Delta t K_s, s + 1, \ldots, \Delta t K_s, n]
\]

because the matrix \( C \) is the diagonal one, and its \( s \)-th diagonal element is equal to \( C_s/\Delta t \).

Here, we apply the Brauer's Theorem which shows the domain of the eigenvalues of any matrix. (6)

\[
1 - a_{ss} \Delta t - P_s \Delta t \leq \lambda_{\text{max}} \leq 1 - a_{ss} \Delta t + P_s \Delta t
\]

where \( a_{ss} = K_{s,s} \) and \( P_s = \sum_{k=1}^{s-1} |K_{s,k}| + \sum_{k=s+1}^{n} |K_{s,k}| \)

Assume that \( A \) is a convergence matrix. Then, its \( \lambda_{\text{max}} \) must be subject to the restrictions of eq.s (2.16) and (2.17).

Therefore,

\[
-1 \leq 1 - a_{ss} \Delta t - P_s \Delta t
\]
\[
1 \geq 1 - a_{ss} \Delta t + P_s \Delta t
\]

From eq. (2.18) we obtain

\[
\Delta t \leq \frac{2}{a_{ss} + P_s}
\]
And, from eq.(2.19) we obtain
\[ (-a_{ss} + P_s) \Delta t \leq 0 \]  \hspace{1cm} (2.21)

Since \( \Delta t \geq 0 \), eq(2.21) yields to
\[ -a_{ss} + P_s \leq 0 \]  \hspace{1cm} (2.22)

But in practice the case
\[ a_{ss} + P_s > 0 \]  \hspace{1cm} (2.23)
exists.

This contradiction appears because \( \lambda_{\text{max}} \) is subject to the restrictions of eq.s (2.18) and (2.19). Therefore, the domain where the real eigenvalues locate should be expressed as following instead of eq.(2.17).

\[ 1 - a_{ss} \Delta t - P_s \Delta t + \beta \leq \lambda_{\text{max}} \leq 1 - a_{ss} \Delta t + P_s \Delta t - \alpha \]  \hspace{1cm} (2.24)

where \( \alpha \) and \( \beta \) are the positive values. Eq.(2.24) is explained by the illustration given in Fig.2-2.

Here, we should notice that eq.(2.24) has no contradiction to the Brauer's Theorem, because the theorem can show only the maximum domain where all eigenvalues exist. Then, \( \lambda_{\text{max}} \) must be subject to the restrictions of eq.s (2.16) and (2.24), so we obtain

\[ -1 \leq 1 - a_{ss} \Delta t - P_s \Delta t + \beta \]  \hspace{1cm} (2.25)
\[ 1 \geq 1 - a_{ss} \Delta t + P_s \Delta t - \alpha \]  \hspace{1cm} (2.26)
From eq. (2.25)

\[ \Delta t \leq \frac{2 + \beta}{a_{ss} + P_s} \]  

(2.27)

From eq. (2.26)

\[ \Delta t \leq \frac{\alpha}{-a_{ss} + P_s} \]  

(2.28)

But, at present we have no information on the value of \( \alpha \) and \( \beta \). Therefore, we set \( \beta = 0 \) and ignore eq. (2.28), and the critical time-step width is represented as

\[ \Delta t \leq \frac{2}{a_{ss} + P_s} \]  

(2.29)

As far as \( \Delta t \) satisfies eq. (2.29), the calculation converges, and also eq. (2.28) is satisfied automatically. In this paper, above equation is used for the theoretical estimation of \( \Delta t_{cr} \). Above consideration on the convergence condition of eq. (2.29) by use of the Brauer's Theorem results in that the condition is actually determined by the values of entries of \( A \), as shown in eq. (2.29). Let the domain of our concern be as shown in Fig. 2-1. Then, it is obvious that the critical value, \( \Delta t_{cr} \), of whole equation is determined by nodal points in \( R \), on \( S_1 \) or on \( S_2 \). Let's denote the minimum values of \( \Delta t \) in \( R \), on \( S_1 \) and on \( S_2 \) by \( \Delta t_{cr}^{(1)} \), \( \Delta t_{cr}^{(2)} \), \( \Delta t_{cr}^{(3)} \), respectively. Then, the critical time-step, \( \Delta t_{cr} \), for whole system is expressed as following,

\[ \Delta t_{cr} = \min ( \Delta t_{cr}^{(1)}, \Delta t_{cr}^{(2)}, \Delta t_{cr}^{(3)} ) \]  

(2.30)

We show that \( \Delta t_{cr} \) has no influence on the convergence condition as far as we use the explicit solution method.

Let's rewrite eq. (2.13) as following,

\[ \{ \ddot{\Phi}_i \} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \{ \ddot{\Phi}_{i-1} \} + \begin{bmatrix} \ddot{B} \\ B_1 \end{bmatrix} \]  

(2.31)

, where subscript "1" indicates the nodal value of \( S_1 \). Since \( \dot{\Phi}_{i,1} = \dot{\Phi}_{i-1,1} \) \((= \ddot{\Phi}_1)\), we may treat following expression instead of eq. (2.13).
\[ \Phi_i = A_{11} \Phi_{i-1} + A_{22} \Phi_1 + \bar{b} \]  

(2.32)

And, the convergence condition is governed only by the maximum eigenvalue of \(A_{11}\), which is the submatrix of \(A\). Then, we may treat eq.(2.33) instead of eq.(2.30).

\[ \Delta t_{cr} = \min \left( \Delta t_{cr}^{(1)}, \Delta t_{cr}^{(3)} \right) \]  

(2.33)

But, we should notice that \(\Delta t_{cr}\) obtained from eq(2.33) can't give the true critical value, because in above theoretical consideration we set \(\beta=0\) and ignore eq(2.28) and get eq(2.29).

In successive section the difference between the true critical and theoretical values are surveyed through a number of numerical experiments.

3. NUMERICAL EXPERIMENTS ON THE CRITICAL TIME-STEP WIDTH

In preceding section the critical time-step width, \(\Delta t_{cr}\), is theoretically obtained by using the Brauer's Theorem, and the considerations on the estimation method clarified that the theoretical value of \(\Delta t_{cr}\) may be an underestimated one. That is, it is expected that the true critical value is larger than the estimated one.

In this section the extension of the estimated value of \(\Delta t_{cr}\) is measured through a number of numerical experiments. Eq.(2.33) for the estimation of \(\Delta t_{cr}\) clarifies that the critical time-step width is influenced by the physical, geometrical and topological properties which decide the values of entries in the coefficient matrix.

But the most important concern for the user of the mixed-type formulation as given in this paper is what kind of factors of the discretization procedures gives influences to \(\Delta t_{cr}\). Finite element models used in this numerical experiments are as following;

The domain is a unit square area, i.e. \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\), and the area is subdivided into \(N\) and \(M\) nodes located along \(x\) and \(y\) axes, respectively. All finite elements have the same property of constant strain, and only the central node of the region is heated. By using above finite element nodes we examine the influences of following factors to true \(\Delta t_{cr}\).
1. The influence of the lengths of the finite elements.
2. The influence of the different boundary conditions.
3. The influence of the combination of several element lengths. Furthermore, by slight modification of above model we can examine following item;
4. The influence of the angle of finite elements.

The results obtained by the numerical experiments are as followings;

[Experiment 1] Uniformly subdivided model

The purpose of this experiment is to know the influence of the element-length on $\Delta t_{cr}$. The unit area is uniformly subdivided, and the lengths of element is $\Delta x = 1/(N-1)$ and $\Delta y = 1/(M-1)$ along x and y axes, respectively. (See Fig.3-1)

![Fig.3-1 Uniformly subdivided model](image_url)

All boundaries is $s_1$ type, because by this restriction the influence of the boundary condition is removed as mentiond in previous section. The theoretical value of $\Delta t_{cr}$ is easily obtained by applying the Brauer's Theorem for an arbitrarily selected row in the coeffi-
cient matrix $A$.

$$1 - 4 \left( \frac{\Delta t}{\Delta x^2} + \frac{\Delta t}{\Delta y^2} \right) \leq \lambda_{\text{max}} \leq 1$$  \hspace{1cm} (3.1)

where $\Delta x$ and $\Delta y$ are the element lengths along $x$ and $y$ axes, respectively. Since we have $|\lambda_{\text{max}}| \leq 1$ for the numerical convergence, then the following critical value of the time-step width is obtained;

$$\Delta t_{\text{cr}} \leq \frac{1}{2} \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$  \hspace{1cm} (3.2)

Above condition is just same as the one which is obtained by FDM applied for spatial and time axes. The results of the numerical experiments are summarized in Table 3-1, and from the table we can remark following items;

1. The theoretical values of $\Delta t_{\text{cr}}$ have good coincidence with the experimental values.
2. The extension of $\Delta t_{\text{cr}}$ becomes large when both of $\Delta x$ and $\Delta y$ are enlarged at the same time.

This experiments result in that eq.(3.2) can give good estimation of the critical value of the time-step width for this case.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>0.025</th>
<th>0.050</th>
<th>0.100</th>
<th>0.167</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>1.561</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$T$</td>
<td>1.561</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$E$</td>
<td>2.500</td>
<td>6.249</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$T$</td>
<td>2.500</td>
<td>6.249</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$E$</td>
<td>2.940</td>
<td>10.27</td>
<td>25.66</td>
<td>-</td>
</tr>
<tr>
<td>$T$</td>
<td>2.940</td>
<td>10.00</td>
<td>25.00</td>
<td>-</td>
</tr>
<tr>
<td>$E$</td>
<td>3.093</td>
<td>11.77</td>
<td>37.90</td>
<td>71.13</td>
</tr>
<tr>
<td>$T$</td>
<td>3.093</td>
<td>11.46</td>
<td>36.80</td>
<td>69.66</td>
</tr>
</tbody>
</table>

($\times 10^{-4}$)

Note; $E$ and $T$ means the experimental and theoretical values, respectively.
[Experiment 2] Irregularly subdivided model

The aim of this experiment is to know the influence of the different element lengths set in a model. For this purpose a unit square with only $s_1$-type boundary condition is irregularly subdivided along x-axis and uniformly along y-axis. The loading condition is just same as the one of the experiment 1. The model used in this experiment is illustrated in Fig.3-2, and the results are summarized in Fig.3-3,3-4 and 3-5.

![Models for the experiments](image)

Fig.3-2 Models for the experiments

![Growth of Δt](image)

Fig. 3-3 A Type
Convergence Condition of Explicit Finite Element Method

Fig. 3-4 B Type

Ratio of edge lengths ($\Delta x_1 : \Delta x_2 : \Delta x_3$)

Growth of $\Delta t$

$\frac{n_1}{n_2} : \frac{n_2}{n_3} = \frac{2}{6}:2$

$3:4:3$

$4:2:4$

Fig. 3-5 C Type

Ratio of edge lengths ($\Delta x_1 : \Delta x_2 : \Delta x_3 : \Delta x_4 : \Delta x_5$)

Growth of $\Delta t$

$\frac{n_1}{n_2} : \frac{n_2}{n_3} : \frac{n_3}{n_4} : \frac{n_4}{n_5} = \frac{2}{2}:2:2:2:2$

$3:1:2:1:3$
The theoretical value given in the figure for the comparison are obtained by applying the Brauer's Theorem for the row, which corresponds to the node surrounded by the smallest elements, in the coefficient matrix $A$. Then, we obtain

$$1 - 4 \left( \frac{\Delta t}{\Delta x_s^2} + \frac{\Delta t}{\Delta y_s^2} \right) \leq \lambda_{\text{max}} \leq 1$$

(3.3)

where $\Delta x_s$ and $\Delta y_s$ are the smallest element lengths, which surround the node, along $x$ and $y$ axes, respectively.

Since we have $|\lambda_{\text{max}}| \leq 1$ for the numerical convergence, then the following critical value of the time-step width is obtained;

$$\Delta t_{cr} \leq \frac{\Delta x_s^2 \Delta y_s^2}{\Delta x_s^2 + \Delta y_s^2}$$

(3.4)

The result of the experiments are summarized as followings;

(1) A node surrounded by finite elements with the smallest element length governs the critical value of $\Delta t$ as it is expected from the theoretical consideration.

(2) The number of finite elements with the largest edge length gives influence to $\Delta t_{cr}$, and their location also gives influence to the value. That is, if they are located at the central area of the domain, then we may expect that true critical value of $\Delta t$ is larger than the theoretical evaluation.

(3) The tendency of the increasing of observed values of $\Delta t_{cr}$ comparing to its theoretical value is strengthened as the domain is discretized irregularly.

[Experiment 3] Angle of triangular finite element

For this numerical experiment the domain is slightly modified and we treat a parallelogram as shown in Fig.3-6 instead of a square area.

For this modification we consider two types of parallelograms (A and B shown in Fig.3-6), and both cases are used for the experiments. Their surrounding edges are restricted to be $s_1$-boundary condition. Theoretical values of $\Delta t_{cr}$ for this experiment are evaluates as followings;
(1) The acute-angled triangular elements (See Fig. 3-7)

The entries of an arbitrarily selected row in the coefficient matrix $A$ is as following:

\[
\begin{bmatrix}
L_1^2 + L_2^2 - L_3^2 \Delta t & -L_1^2 + L_2^2 + L_3^2 \Delta t \\
8\Delta^2 & 8\Delta^2 \\
\end{bmatrix}
\begin{bmatrix}
L_1^2 - L_2^2 + L_3^2 \Delta t & \frac{L_1^2 + L_2^2 + L_3^2 \Delta t}{4\Delta^2} & \frac{L_1^2 - L_2^2 + L_3^2 \Delta t}{8\Delta^2} \\
8\Delta^2 & 4\Delta^2 & 8\Delta^2 \\
\end{bmatrix}
\begin{bmatrix}
L_1^2 - L_2^2 + L_3^2 \Delta t & \frac{L_1^2 + L_2^2 - L_3^2 \Delta t}{4\Delta^2} \\
8\Delta^2 & 4\Delta^2 & 8\Delta^2 \\
\end{bmatrix}
\]

where $L_1, L_2, L_3$ are the length of three edges of the triangular element shown in Fig. 3-5 and $\Delta$ is the area.

Then, the Brauer's Theorem gives following expression;

\[
1 - \frac{L_1^2 + L_2^2 + L_3^2}{2\Delta^2} \Delta t \leq \lambda_{\text{max}} \leq 1
\]  

(3.5)

Consider that $\lambda_{\text{max}}$ must locate between -1 and 1, then we obtain following stability condition for $\Delta t_{cr}$:

\[
\Delta t_{cr} \leq \frac{4\Delta^2}{L_1^2 + L_2^2 + L_3^2}
\]  

(3.6)

(2) The obtuse-angled triangular elements (See Fig. 3-8)

The entries of a row in the coefficient matrix $A$ is the same one of the first case. The application of the Brauer's Theorem yield to be

\[
\left| \lambda_{\text{max}} - (1 - \frac{L_1^2 + L_2^2 + L_3^2}{4\Delta^2} \Delta t) \right| \leq \frac{-L_1^2 + 3L_2^2 - L_3^2}{4\Delta^2} \Delta t
\]  

(3.7)

where $L_1, L_2$ and $L_3$ are lengths of three edges of the triangle element and we assume that $L_2$ is the longest one among them.

The same procedure presented in case (1) follows to

\[
\Delta t_{cr} \leq \frac{2\Delta^2}{L_2^2}
\]  

(3.8)
The edge lengths are invariable.

Fig. 3-6 The parallelogram domain

The acute-angled triangular element

Fig. 3-7

The obtuse-angled triangular element

Fig. 3-8

Now, we show the results of the numerical experiments.

(See Fig. 3-9)

From these experiments we can notice following items:

(1) The increasing of the observed value of $\Delta t_{cr}$ to the theoretical one is maximized at $\theta = 60^\circ$ in both cases, A and B, and the growth of $\Delta t$ for the theoretical value reaches round 30%.

(2) For case B with $\theta < 45^\circ$ we can't expect the increasing of the critical time-step width comparing to the theoretical one.

(3) From the first item the discretization using regular triangle
finite elements is preferable from the viewpoint of the numerical stability, because increasing of $\Delta t_{cr}$ is expected.

![Graphs showing growth of $\Delta t$ for Type A and Type B](image)

**Fig. 3-9** The results of Experiment 3

[Experiment 4] Boundary condition

In the mixed-type formulation given in section 2 we consider two type of boundary conditions, i.e. $S_1$ and $S_2$. The consideration in previous section clarified that $S_1$-condition gives no influence to the numerical stability. Therefore, in this experiment we consider the influence of $S_2$-boundary condition, that is

$$\frac{\partial \phi}{\partial n} = -h(\phi - v)$$

(3.9)

where $h$ is the heat transfer coefficient and $v$ is the value of $\phi$ out of the region.

For this purpose the square area is regularly subdivided, and for all nodes on boundary above boundary condition is applied. This boundary condition is introduced in the coefficient matrix as the matrix $B_1$ and $B_2$ in eq.(2.12). In this experiment we must examine the propriety of eq.(2.33). $\Delta t_{cr}^{(1)}$ shown in eq.(2.33) is obtained as eq.(3.2). $\Delta t_{cr}^{(3)}$ is obtained as following.

The entries of the row, which correspond the node on $S_2$, are as
followings;
\[
\frac{3 \Delta t}{2 \Delta x^2} \cdots \frac{3 \Delta t}{2 \Delta y^2} - \frac{1 \Delta t}{2} - \frac{h}{\lambda} \left[ \frac{1}{2 \Delta x^2} + \frac{1}{\Delta y^2} + \frac{h}{\lambda} \right] \Delta t
\]
\[
= \frac{3 \Delta t}{2 \Delta y^2} - \frac{1 \Delta t}{2} - \frac{h}{\lambda} \Delta t
\]
\[
\frac{3 \Delta t}{2 \Delta y^2} - \frac{1 \Delta t}{2} - \frac{h}{\lambda} \Delta t \quad \cdots \quad \text{...}
\]
\[
\text{where } \lambda \text{ is the element length on } s_2.
\]

Then, the Brauer's Theorem gives following expression;
\[
1 - \frac{\Delta t}{\Delta x^2} - \frac{\Delta t}{\Delta y^2} - \frac{h}{\lambda} \leq \lambda_{\text{max}} \leq \frac{\Delta t}{1 - 3h/\lambda}
\]
(3.10)

Consider that \( \lambda_{\text{max}} \) must locate between -1 and 1, then we obtain
\[
\Delta t \leq \frac{2}{3} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{h}{\lambda} \right)
\]
(3.11)
\[
3 \frac{\Delta t}{\lambda} \geq 0
\]
(3.12)

In case \( h > 0 \), eq.(3.12) is absolutely satisfied. So we obtain
\[
\Delta t_{\text{cr}}^{(3)} \leq \frac{2}{3} \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{h}{3\lambda} \right)
\]
(3.13)

<table>
<thead>
<tr>
<th>Table 3-2 The result of Experiment 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0.1</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>10.0</td>
</tr>
<tr>
<td>100.0</td>
</tr>
</tbody>
</table>
The results are summarized in Table 3-2, and the table indicate following items:

(1) The physical value "h" gives influence to $\Delta t_{cr}$.

(2) The more the physical value "h" increase, the bigger the real $\Delta t_{cr}$ is influenced by $\Delta t_{cr}$ (3), that is, it is clarified the boundary condition shown in eq.(3.9) influence the convergence strongly.

4. CONCLUDING REMARKS

In this paper we presented the discretization method of the Heat Transfer Equation by using the finite element method in spatial axes and the explicit difference scheme in time axis, and the convergence condition of above explicit equation is surveyed theoretically and experimentally. The theoretical approach could give several convergence conditions by applying the Brauer's Theorem, and through numerical experiments the propriety of proposed convergence conditions is examined. At the same time, it was also remarked that the critical time-step width by proposed method is underestimated one, and that the difference between the true critical width is governed by edge-length, angle of triangular finite elements, their rearrangement in the problem area and boundary condition. The authors could clarify that this difference is caused by the reduction process of the theoretical estimation method using the Brauer's Thorem. Generally speaking, the smallest finite element governs the critical time-step width, $\Delta t_{cr}$, except the boundary condition, and for the numerical stability the finite element model should consist of only regular triangular meshes.

References

4). Ditto of 1), 103-108
6) Ditto of 2), 68-70