Behaviors of a Soliton in Nonlinear L-C Lines with Abrupt Parameter Change

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(Received November 2, 1979)

Synopsis

Some behaviors of a lattice soliton in nonlinear L-C ladder lines with abrupt parameter change are investigated. The results of computer and circuit experiments show that in the case of a soliton incident upon the line of larger characteristic impedance and of higher phase velocity in linearized-line-limit, the transmitted wave evolves into larger number of solitons. The experimental results can be well explained by use of both linearization approximation for the line near the junction and the theory by Gardner, Greene, Kruskal and Miura.

1. Introduction

Recently much attention has been paid to "soliton" in various fields of science and technology because of its interesting and peculiar properties [1]. Soliton is a pulse-like solitary wave which can exist in nonlinear dispersive media and, due to the contribution by R. Hirota and K. Suzuki can easily observed in nonlinear L-C ladder line [2]-[5].

The soliton is found originally as a stationary solitary wave solution of the constant-coefficient Korteweg-de Vries equation [6] which describes wave propagation in homogeneous media with nonlinearity and dispersion. But actual media usually have some inhomogeneities, and regarding their effects on the soliton propagation, considerably

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many studies have been made to investigate the effects of gradual parameter change in medium through computer experiments and analyses by use of perturbation method for several sorts of waves: shallow water waves [7]-[9], nonlinear lattice waves [10], and magneto-acoustic waves in plasma [11]. On the other hand, there seems to be only a few reports for the effects of abrupt parameter change [12] [13].

In this paper, some behaviors of a lattice soliton in nonlinear L-C ladder lines with abrupt parameter change are studied. Computer experiments are performed on the time evolution of a soliton incident on a junction of two uniform lines with different parameter values, and the results are interpreted by use of both linearization approximation for the line near junction and Gardner-Green-Kruskal-Miura theory [14]. Experiments using electrical network are also carried out and compared with the computer results.

2. A lattice soliton on a nonlinear ladder line

Consider a nonlinear L-C ladder line as shown in Fig.1. The line

![Diagram of L-C ladder line](image)

is composed of linear inductors of inductance L and nonlinear capacitors whose differential capacitance $C(V)$ is given as a function of applied voltage $V = V_0 + V_n$ by

$$C(V) = C_0 \left[ 1 + \frac{V_n}{V_0} \right],$$  \hspace{1cm} (1)

where $V_0$ is the bias voltage.

The line equation are written, from Fig.1, as
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\[ V_{n+1} - 2V_n + V_{n-1} = \frac{d}{dt} \left[ C(V_n) \frac{dV_n}{dt} \right] \]  

From Eqs. (1) and (4), we obtain a stationary solitary wave solution such that

\[ V_n - V_\infty = V^* \text{sech}^2 (\beta^* t - \alpha n), \]  

where

\[ \beta^* = \frac{1}{\sqrt{LC_0}} \sqrt{\frac{1}{1 + \frac{V_\infty}{V_0}} \sinh \alpha}, \]  

\[ V^* = LC_0 \beta^* \beta^2 V_0, \]  

and

\[ V_\infty = \lim_{n \to \infty} V_n. \]  

Eq. (5) expresses a waveform as shown in Fig. 2, and is called a lattice soliton [15]. As Eq. (5) shows that a soliton travels at a velocity of \( \beta^*/\alpha \) sections per unit time, it can be seen from Eqs. (6) and (7) that the larger amplitude has a soliton, the faster it travels. It is also well-known that solitons pass through with each other without losing their identities in spite of the wave propagation on a nonlinear line [16].

Now, if the line is not uniform, steady propagation in the form of a soliton will be impossible. In such case, an incident soliton may split into two or more solitons, or may disintegrate into a train of ripples.
3. **Behaviors of a soliton on a line with abrupt parameter change**

As a simple case of a line with spatially varying parameters, we consider a line which is composed of two parts with different parameter value from each other (see Fig.3). Let us denote the first part of the line as Line-1, and the second part as Line-2, and be the parameters of each line denoted as $L_i$ and $C_i$ with $i=1,2$.

Suppose that a soliton propagating on the Line-1 is expressed by

$$S = \frac{c_{10}}{1 + \frac{V_n^2}{V_0^2}} \text{sech}^2 (\beta t - \alpha n),$$

$$\beta = \sinh \alpha, \quad \tau = t/\sqrt{L_1 C_01}.$$  \hspace{1cm} (9)

Our concern is what will be the time evolution of the soliton after passing the boundary of Line-1 and Line-2. Some results of computer experiments on this problem are shown in Fig.4, for the case $\alpha = 0.5$. In the first two of the figure, (a) and (b), linearized propagation velocity is equal on the both line. In (a), the linearized characteristic impedance of Line-2, $Z_{02}$, is small than that of Line-1, $Z_{01}$, and reflection at reversed polarity results as in the usual transmission line. In this case, the transmitted wave of considerably small amplitude disintegrates into one soliton and ripple. In (b), $Z_{02}$ is twice as large as $Z_{01}$ and it is seen that a wave of slightly larger amplitude is generated just after passing the boundary, which in turn is evolved.
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Fig. 4 Time evolution of a soliton incident on the junction (J) of two uniform lines.

Into one large soliton and another small soliton accompanied by an oscillating trail. In the last two of Fig. 4, (c) and (d), the input wave to the Line-2 is broadened. In (c), the input wave has slightly larger amplitude and disintegrates into two solitons and ripple. In (d), the transmitted wave has sufficiently width so that more than
three solitons are generated through its time evolution. We are much interested especially in the case (d), in which both lines are almost perfectly matched as in the linear problem, and incident disturbance is broadened only because of the increase in propagating velocity.

4. Consideration for the transmitted wave

Let us now introduce continuum approximation for the L-C line shown in Fig.1 under the assumption of small nonlinearity and small dispersion. If $V_n$'s have gradual spatial variation, we can expand $V_{n\pm 1}$ in a Taylor's series:

$$V_{n\pm 1} = V_n + \frac{D}{2} \frac{\partial V}{\partial x} + \frac{D^2}{12} \frac{\partial^2 V}{\partial x^2} + \frac{D^3}{3!} \frac{\partial^3 V}{\partial x^3} + \frac{D^4}{4!} \frac{\partial^4 V}{\partial x^4} + \cdots, \quad (11)$$

where, denoting the spatial length of the line per section as D,

$$x = nD \quad (12)$$

and

$$V(x,t) = V_n(t). \quad (13)$$

In the case of $V_n/V_0 < \eta < 1$, where $\eta$ is a constant, Eq.(1) is written in the form,

$$C^2 \left( \frac{V_n}{V_0} \right) = \frac{V_n}{V_0} + \delta f \left( \frac{V_n}{V_0} \right), \quad (14)$$

where $\delta$ is a small parameter. Substitution of Eqs.(11) and (14) into Eq.(4) gives

$$\frac{\partial^2 V}{\partial x^2} + \frac{D^2}{12} \frac{\partial^4 V}{\partial x^4} + \frac{LC_0}{D^4} \frac{\partial}{\partial t} \left( 1 - V \frac{\partial V}{\partial t} \right) = 0 \left( D^4, \delta \right), \quad (15)$$

where $V = V/V_0$.

Let us introduce the co-ordinate transformation such as,
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\[ \xi = x - \frac{D}{\sqrt{LC_0}} t \]
\[ T = \epsilon t \]

(16)

where \( \epsilon \) is a small parameter relating to the wave number. Substituting Eq.(16) into Eq.(15) and neglecting \( 0(\epsilon, D^4/\epsilon, \delta/\epsilon, \eta) \), we obtain

\[ \frac{\partial v}{\partial T} + av \frac{\partial v}{\partial \xi} + b \frac{\partial^3 v}{\partial \xi^3} = 0, \]

(17)

where

\[ a = \frac{D}{2\epsilon \sqrt{LC_0}} \]

(18)

and

\[ b = \frac{D^3}{24\epsilon \sqrt{LC_0}}. \]

Eq.(17) is the Korteweg-de Vries (KdV) equation. Putting

\[ u = -v \]

\[ \xi_2 = \frac{a}{b} \xi \]

(19)

\[ T_2 = \left( \frac{a}{b} \right)^2 \cdot \frac{1}{b} \cdot T \]

Eq.(17) can be rewritten as

\[ \frac{\partial u}{\partial T_2} - 6u \frac{\partial u}{\partial \xi_2} + \frac{\partial^3 u}{\partial \xi_2^3} = 0. \]

(20)

Gardner, Greene, Kruskal and Miura (GGKM) showed that if \( u(\xi_2, T_2) \)
varies with time according to Eq.(20), the eigenvalues \( E_m \) \( (m=0,1,2,\cdots,N-1) \) for the time-independent Schrödinger equation are time invariant \([14]\) :

\[
\frac{\partial^2 \psi(m)}{\partial \xi_2^2} - [u(\xi_2, T) - E_m] \psi(m) = 0.
\]

Thus, the bound-state eigenvalues \( E_m (<0) \) are preserved and are associated with the amplitudes \( A_m \) of the solitons that are produced from an arbitrary initial wave \( u(\xi_2, 0) \) as

\[
A_m = 2E_m.
\]

If the initial wave is given by

\[
v(\xi, 0) = A^* \text{sech}^2(\xi/l)
\]

in the co-ordinate \((\xi,T)\), the eigenvalues for the corresponding Schrödinger equation are easily found \([17]\), and the amplitudes of the produced solitons are determined. The result is that, putting

\[
\frac{a}{6b} A^* l^2 = p(p+1),
\]

the amplitudes of the produced solitons, \( A_k \)'s, are

\[
A_k = \frac{2A^*}{p(p+1)} (p-k)^2,
\]

\[
(k=0,1,2,\cdots,N_p-1)
\]

where \( N_p \) is the largest integer satisfying the inequality \( N_p < p+1 \). The initial wave is disintegrated into \( N_p \) solitons or \( N_p \) solitons accompanied by ripple according as \( p \) is integer or not. Fig.6 shows the relation between \( p^2 \) and normalized amplitudes \( A_k/A^* \) of produced solitons.

Next, we must obtain the values of \( p \) for incoming wave to Line-2, that is, for the initial perturbation. Denote the distributed-line-limit of the characteristic impedance of Line-1 and Line-2 when linearized as \( Z_{01} \) and \( Z_{02} \), respectively. Then the transmission
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The coefficient from Line-1 to Line-2 is given as

$$T = \frac{2z_{02}}{z_{01} + z_{02}}, \quad (26)$$

and the amplitude of the transmitted wave into Line-2, $A_{20}^*$, is

$$A_{20}^* = T A_{10}^* = \frac{2}{1 + \sqrt{\sigma_c/\sigma_L}} A_{10}^*, \quad (27)$$

where $A_{10}^*$ is the amplitude of the incident soliton and

$$\sigma_L = \frac{L_2}{L_1}, \quad \sigma_c = \frac{C_{02}}{C_{01}}. \quad (28)$$

As the propagation velocity can be approximated by $1/\sqrt{\sigma_L \sigma_c}$ ($i=1,2$) on each line, the width of the transmitted wave into Line-2, $\xi_2$, can be related to that of the incident soliton on Line-1, $\xi_1$, by

$$\xi_2 = \frac{L_1 C_{01}}{L_2 C_{02}} \xi_1 = \frac{1}{\sqrt{\sigma_L \sigma_c}} \xi_1. \quad (29)$$

Let us denote the $p$ values for the incident soliton and transmitted wave as $p_1$ and $p_2$, respectively. Since $p_1 = 1$ in the continuum approximation, Eq.(24) gives

$$\frac{a}{6b} A_{10}^* \xi_1^2 = 2. \quad (30)$$

Assuming that the shape of the wave which has just entered into Line-2 is almost the same as a soliton, we have from Eq.(24)

$$\frac{a}{6b} A_{10}^* \xi_2^2 = p_2 (p_2 + 1). \quad (31)$$

From Eqs.(27), (29), (30), and (31), we obtain

$$p_2 (p_2 + 1) = \frac{4}{\sigma_c \sigma_L (1 + \sqrt{\sigma_c/\sigma_L})}. \quad (32)$$
For given \( \sigma_c \) and \( \sigma_L \), \( p_2 \) is determined by Eq. (32). The number of solitons produced from the transmitted wave is equal to the largest integer included in \( p_2 + 1 \), and the individual soliton has the amplitude given by inserting \( A = A_{20} \) and \( p_2 \) into Eq. (25). A contour map for \( p_2 \) in \( \sigma_c - \sigma_L \) plane is given in Fig. 5. The results of the computer experiments are shown in Fig. 6, where the amplitudes of the produced solitons normalized by \( A_{20} \) of Eq. (27) are plotted for eleven values of \( (\sigma_c, \sigma_L) \) which are shown in Fig. 5, for the case \( \alpha = 0.5 \). Fig. 6 shows that the values of the computer experiments agree well with the calculated results.

For an incident soliton of large amplitude, the nonlinearity and dispersion is so strong that the line equation (4) can not be led to the Korteweg-de Vries equation such as Eq. (17). The results of computer experiments is plotted in Fig. 6 for the case that the incident soliton has a phase constant \( \alpha = 1.5 \), therefore a normalized amplitude \( \sinh^2 \alpha = 4.53 \). It can be seen from Fig. 6 that the values of the computer experiments agree fairly well with the calculated results obtained from Eqs. (27), (29), and (32) even in such strong nonlinear

Fig. 5 Contour map for \( p \).
and dispersive cases. Agreement is relatively good when $Z_{02}/Z_{01} \approx 1$. These results indicate the possibility that the approximate calculation justifiable for weak nonlinearity and dispersion is also applicable to the case of relatively strong nonlinearity and dispersion.

5. Experiments

Experimental confirmation of some results obtained in the previous sections was carried out using a circuitry as shown in Fig. 7. The
inductance $L_1$ is 39 $\mu$H with error of at most 10 percent. The voltage-dependent capacitance is $C_1(V) = 108(1+\frac{V}{V_0})^{-1}$ pF with error of at most 5 percent, where $V=V_0+V_1$ is the applied voltage and $V_0=4$ volt is the bias voltage. The $Q$ of the inductor is almost 115, while the capacitor $Q$ is almost 130 at 2 MHz. The inductance $L_2$ and the capacitance $C_2$ were obtained by making series or parallel combinations of $L_1$ and $C_1$, respectively. The pulse generator provides the L-C line with a single pulse which resembles a soliton.

Some of the experimental results on behaviours of a soliton traveling on a line composed of two parts with different parameter values are shown in Figs. 8 and 9, for the case ($L_2/L_1=2.0$, $C_{02}/C_{01}=0.5$) and ($L_2/L_1=0.5$, $C_{02}/C_{01}=0.5$), respectively. It is seen from these both figures that the width of the wave just after passing the boundary is equal to that of the incident soliton, because these figures represent the time variation of voltage at each section, while Fig.4 represents the spatial variation of voltage. Figs.18(d) and 19(d) show that the transmitted wave evolves to produce one soliton and two solitons, respectively. In the both figures, so small solitons cannot be found because they are masked by small reflective waves caused by irregular variation of the line parameter.

![Fig. 8](image_url)

Fig. 8 Time evolution of a soliton incident on a junction (at n=50) of two lines for the case $L_2/L_1=2.0$ and $C_{02}/C_{01}=0.5$. 

(a) $n=47$
(b) $n=51$
(c) $n=100$
(d) $n=225$
6. Conclusions

Behaviors of a soliton in nonlinear L-C ladder lines with discontinuity in parameter values have been investigated both by computer and circuit experiments, and given a theoretical interpretation.

Typical results obtained include the followings. When a soliton passes a junction with the second line of larger characteristic impedance and of higher phase velocity in linearized-line-limit, it evolves to disintegrate generally into larger number of solitons. Whatever parameter values has the second line, the transmitted wave produce one or more solitons ultimately. These experimental results can be well explained by using both linearization approximation for the line near the junction and Gardner-Greene-Kruskal-Miura theory for the K-dV equation which describes the continuum limit of nonlinear
ladder line. Computer experiment has also shown that the same idea of explaining results is also available for a soliton of large amplitude, though the linearization and the continuum approximation can not be validated for such case.

Further interesting problems to be studies will be 1) similar investigation for the case of inhomogeneity in nonlinearity of the capacitor, 2) time evolution of the reflected wave produced at the discontinuities, and 3) propagating characteristics of a soliton incident on a junction of several lines.

References

[12] F. Yoshida and T. Sakuma, "Computer-simulated scattering of lattice solitons from a mass interface in a one-dimensional nonlinear
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