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Spectrum of Schottky Noise in Ion Storage Rings

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SYNOPSIS

The spectrum of Schottky noise in ion storage rings is analyzed as density fluctuations in effectively one-dimensional plasmas. Strong coupling effects in these plasmas are discussed in relation to experimental observations.

1. Introduction

Storage rings of heavy or light ions are now becoming one of powerful tools for experiments in the field of atomic physics: When ions are sufficiently cooled down by electron cooling and possibly by laser cooling, longitudinal temperatures of the order of meV are attained and very precise results can be obtained for various atomic processes. These ions with very low temperatures also provide us with a chance to verify the properties of strongly coupled plasmas which have been predicted by numerical experiments.

Along with ion clouds in ion traps, such as Penning and Paul traps, ions in ion storage rings may be regarded as typical examples of the one-component plasma (OCP). The harmonic confining (restoring) force due to alternating quadrupole magnetic fields is equivalent to the existence of the uniform background of opposite charges

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around the center of the beam.

The most remarkable phenomenon resulting from strong coupling in these ion beams may be crystallization. Crystallization in cold ion systems with finite extension has been first shown by numerical experiments[1] and then observed as the shell structure of laser cooled ions in the Penning trap.[2] These structures have been theoretically analyzed by Barrat and the author and positions and populations of these shells have been successfully reproduced by a simple model which takes into account the effect of discreteness or correlation as an essential part of the theory.[3,4,5]

For cold ion beams in ion storage rings, the crystallization has not been confirmed; the transverse temperature is of the order of eV and is much higher than the longitudinal one. The maximum value of the coupling parameter Γ hitherto reported is around 5, while the crystallization is expected for $\Gamma \sim 180$.[6]

As for dynamic properties, the macroscopic deformation of ion clouds in the Penning trap has been observed[7] and given a theoretical explanation.[8] Microscopic charge density fluctuations in ion beams have been observed as the Schottky noise spectrum. It has been reported that the electron cooling or the increase of the number of ions has strong effects on these spectrum even before crystallization; the appearance of the collective mode and the reduction of noise power (the integral of frequency spectrum) have been observed.[9]

The purpose of this paper is to analyze the density fluctuation in ion beams and to discuss the effect of strong coupling on these spectra.

2. Ion beam as One-Dimensional Plasma

We consider an ion beam of radius a circulating in an ion storage ring of radius R. We regard the beam as straight in the z-direction, while imposing the periodic boundary condition with the period $2\pi R$ in this (longitudinal) direction. We denote the charge of an ion by q and the transverse coordinates by $\mathbf{r}$.

When the existence of the grounded vacuum chamber of radius b is taken into account, the coulomb energy of the beam with the number density $\rho(r,z)$ is written as

$$H = \frac{1}{2} q \int d\mathbf{r} dz \phi(r,z) \rho(r,z), \tag{1}$$

where $\phi(r,z)$ is the electrostatic potential given by

$$\phi(r,z) = \frac{1}{2\pi R} \sum_{k_z} \frac{1}{2\pi} \int dkk \frac{4\pi q}{k^2 + k_z^2} \rho_{k,k_z} [J_0(kr) - J_0(kb) \frac{I_0(k_z r)}{I_0(k_z b)}] \exp(ik_z z), \tag{2}$$

$$\rho_{k,k_z} = \int d\mathbf{r} dz \rho(r,z) J_0(kr) \exp(-ik_z z). \tag{3}$$

Here $J_0(x)$ and $I_0(x)$ are the Bessel and the modified Bessel functions and $k_z = 2\pi l / 2\pi R = l/R$, l being an integer, is the wave number along the axis: l is called the harmonic number.

We now assume that charges in the beam is uniformly distributed in the radial direction as

$$\rho(r,z) = \frac{1}{\pi a^2} \theta(a-r) \rho(z), \tag{4}$$

$$\int dz \rho(z) = N, \tag{5}$$

where N is the total number of ions. After integrating over transverse coordinates, we have H in the form

$$H = \frac{1}{2} \frac{1}{2\pi R} \sum_{k_z} V(k_z) \mid \rho_{k_z} \mid^2, \tag{6}$$

with

$$V(k_z) = \frac{4q^2}{(k_z a)^2} [1 - 2I_1(k_z a) K_1(k_z a) - 2I_1^2(k_z a) \frac{K_0(k_z b)}{I_0(k_z b)}] \tag{7}$$

and

$$\rho_{k_z} = \int dzd\mathbf{r}\rho(r,z)\exp(-ik_z z) = \int dz\rho(z)\exp(-ik_z z) \tag{8}$$

Here $I_1(x)$, $K_0(x)$, and $K_1(x)$ are the modified Bessel functions. We thus have an effectively one-dimensional plasma with a modified Coulomb interaction $V(k_z)$.

When $k_z a \gg 1$, the modified interaction reduces to the one-dimensional Coulomb interaction between charged sheets as

$$V(k_z) \sim \frac{4q^2}{(k_z a)^2}. \tag{9}$$

This one-dimensional Coulomb interaction is cut off at $k_z a \sim 1$ by the finiteness of the beam radius; when $k_z a \ll 1$

$$V(k_z) \sim \left[2\ln(\frac{b}{a}) + \frac{1}{2}\right] q^2. \tag{10}$$

The effective interaction $V(k_z) = V(l)$, regarded as a function of the harmonic number, is equivalent to the impedance for the beam except for trivial factors.[10]

In what follows, we are interested only in z or k_z dependence of various quantities and denote k_z by simply k.

The Schottky noise expresses the fluctuation of electrostatic potential near the beam. Its spectrum is thus directly related to the charge density fluctuation in the beam which is characterized by static and dynamic form factors.

The static form factor $S(k)$ is defined by

$$S(k) = \frac{1}{N} < \rho_k(t)\rho_{-k}(t) > . \tag{11}$$

Here $<>$ denotes the statistical average and the time dependence is made explicit for ρ_k. For a random distribution, $S(k) = 1$. This is also the limiting value for large values of k.

The dynamic form factor $S(k,\omega)$ is defined by

$$S(k,\omega) = \frac{1}{2\pi}\int dt \exp(i\omega t) < \rho_k(t)\rho_{-k}(0) > . \tag{12}$$

These two form factors are related by

$$S(k) = \frac{1}{N}\int d\omega S(k,\omega). \tag{13}$$

The Schottky noise spectrum and its integral over the frequency are proportional to the dynamic and the static form factors, respectively. Note that when the average velocity of ions is v_0, we have to shift the frequency from ω to $\omega - kv_0 = \omega - l\omega_0$ where $\omega_0 = v_0/R$.

Since our system is classical, the fluctuation-dissipation theorem relates the dynamic form factor to the dielectric response function $\epsilon(k,\omega)$ as

$$S(k,\omega) = -\frac{k_B T}{\pi\omega V(k)} 2\pi R \Im \frac{1}{\epsilon(k,\omega)}, \tag{14}$$

where T is the temperature and $\Im$ denotes the imaginary part. Due to (13), the static form factor is then given by

$$S(k) = \frac{k_B T}{nV(k)}\left[1 - \frac{1}{\epsilon(k,\omega=0)}\right], \tag{15}$$

where n is the average number density

$$n = \frac{N}{2\pi R}. \tag{16}$$

3. Random Phase Approximation

The screened response function $\chi_{sc}(k,\omega)$ is defined as the response of the density to the total potential field, the potential due to the sum of external and induced charge densities. The dielectric response function is related to the screened density response function $\chi_{sc}(k,\omega)$ by

$$\epsilon(k,\omega) = 1 - V(k)\chi_{sc}(k,\omega) \tag{17}$$

In the random phase approximation (RPA) or the mean field approximation, the screened response function is replaced by the response of the ideal gas $\chi^0(k,\omega)$ or the Vlasov expression as[10,11]

$$\epsilon(k,\omega) = 1 + V(k)\int dp \frac{k(df/dp)}{\omega - kp/m} \tag{18}$$

Here m is the mass of an ion and $f(p)$ is the distribution function for momentum normalized as

$$\int dp f(p) = n. \tag{19}$$

As in the case of ordinary plasmas, RPA is expected to give an appropriate description of collective properties of this system in the domain of weak coupling.

From (15) and (18), the static form factor is given as

$$S(k) = \frac{1}{1 + nV(k)/k_BT}. \tag{20}$$

Here we have used that $f(p)$ is Maxwellian in thermal equilibrium.

The reduction of the integrated Schottky noise spectrum results from this expression for $S(k)$:[10,11] When the number of ions is sufficiently large or the temperature is sufficiently low, $nV(k)/k_BT$ becomes comparable with unity and we have decreased values for the static form factor for a given harmonic number (or k), as observed in experiments.[9]

The solution of the equation

$$\epsilon(k,\omega) = 0 \tag{21}$$

gives the collective mode. For long wavelengths, the dielectric response function (18) is expanded into an asymptotic series as

$$\epsilon(k,\omega) \sim 1 - \frac{nV(k)}{m\omega^2}k^2. \tag{22}$$

The dispersion relation of the collective mode is thus given by[10,11]

$$\omega(k) = \pm \left[\frac{nV(k)}{m}\right]^{\frac{1}{2}} k. \tag{23}$$

Since $V(k)$ reduces to a constant for small k, we have a phonon-like dispersion relation. The sound velocity is proportional to the square root of the total number of ions N. This is in agreement with experimental observations.

The imaginary part of the frequency of the collective mode, the Landau damping, is proportional to $df(\omega/k)/dp$ and We have sufficiently small damping when

$$\omega/k \gg \text{thermal velocity}. \tag{24}$$

The cooling of ions thus leads to sharp peaks for collective modes in the Schottky spectrum: This behavior has also been observed by experiments.

We here note that the condition of the well defined collective mode (24) is also the condition for the reduction of the total noise power.

4. Beyond Random Phase Approximation

Let us now consider the effects of strong coupling which are not, at least in principle, described by the RPA. Our main concern is the behavior of the Schottky noise spectrum.

In this kind of analyses, sum rules sometimes provides us with some useful exact relationships between various quantities.

The compressibility sum rule relates the long wavelength limit of the screened response function to the compressibility of the system as[12]

$$\lim_{k\to 0} \chi_{sc}(k, \omega = 0) = -n \left(\frac{\partial n}{\partial P}\right)_T . \tag{25}$$

Here P is the pressure. From this relation, we have, instead of the RPA expression (20),

$$S(k) = \frac{1}{\frac{1}{k_B T}\left(\frac{\partial P}{\partial n}\right)_T + \frac{nV(k)}{k_B T}}. \tag{26}$$

When our system is weakly coupled and the pressure is given by

$$P = nk_B T, \tag{27}$$

this result reduces to the RPA value (20). The deviation of the pressure from the ideal gas value thus appears in the long wavelength behavior of the noise power.

Since the density response function is the Laplace transform of the Poisson bracket between spatial Fourier components of density, the high frequency behavior of the dielectric response function is determined by the equal time correlation function of time derivatives of density fluctuations.[13]

The simplest of such sum rules may be the f-sum rule which concludes the same form for $\epsilon(k,\omega)$ as (22) *even for strong coupling.* The dispersion relation of the collective oscillation therefore may not be affected by the strength of coupling in the long wavelength domain. Since the terms proportional to ω^{-4} and higher orders include the effect of correlation, the dispersion relation in short wavelengths will depend on correlation through terms higher order in the wave number.

As for the static properties, it is naturally expected that $S(k)$ oscillates as a function of k before approaching unity for $k = \infty$.[14] This comes from liquid or solid like ordering of ions in longitudinal direction: The peak positions are at integral multiples of the inverse of the mean distance between ions. For the harmonic number corresponding to these peaks of the static form factor, the integrated noise power increases due to strong coupling, in contrast to the reduction in the weak coupling domain described by RPA.

The observation of the enhancement of the noise power, however, seems to be difficult even for ion beams with sufficiently low temperatures. On the one hand, $N \sim 10^7$ or more may be necessary to have large enough noise for observation. On the other hand, the maximum of the observable harmonic number may also be limited and the corresponding wavelength is much larger than the mean ion spacing.

When ions are strongly coupled, the frequency of the collective oscillation approaches that of phonons in solids: The range of well defined mode also extends to short wavelengths comparable with the inverse of mean ionic distance.[14] The observation of this behavior may also be difficult for the same reason described above.

5. Concluding Remarks

The nature of the spectrum of the Schottky noise in ion storage rings is analyzed by regarding the beam of cold ions as one-dimensional plasma. The effects of finite beam radius and surrounding grounded vacuum tube are properly taken into account and long wavelength behavior of the spectrum is derived including the case where ions are strongly coupled. The analysis of the pressure in the domain of strong coupling is in progress and its effect on the static form factor will be reported elsewhere.

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