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One-Dimensional Classical Plasmas in Ion Traps, Ion Storage Rings, and Semiconductor Quantum Wires

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As plasmas with extremely reduced dimensionality, properties of one-dimensional classical plasmas are analyzed in the domain of strong coupling and static and dynamic structure factors and the plasmon dispersion relation are obtained. These plasmas may be realized in Penning traps with sufficiently strong confinement and also in semiconductor quantum wires under appropriate conditions.

I. INTRODUCTION

One of key observations to understand various phenomena in Penning traps may be that charged particles in the trap can be regarded as embedded in the uniform background of opposite charges.[1] When we neglect correlations between particles, we thus have a charged fluid of finite extension. Normal modes of oscillations of these finite plasmas in the fluid limit and related properties enable us to diagnose physical parameters of these plasmas.[2] Finiteness of extension also leads to novel phenomena not observed in infinite plasmas.

When the correlations or the discreteness of charges is taken into account, we have shell structures, two-dimensional lattices on shells, and inter-shell correlation of positions of charges.[3, 4, 5, 6] The inter-shell correlation may be neglected as a first approximation and we can reproduce shell structure observed in real and numerical experiments by considering the intra-shell cohesive energy and inter-shell electrostatic energy.[7] In these cases, the aspect of plasmas with limited dimensionality becomes clear.

By introducing an extra electrode on the central line of the trap, we can control the structure of confined charges. Especially, we can produce purely two-dimensional system of charges by such a method.[8]

In this paper, we consider classical plasmas with smallest degrees of freedom, the one-dimensional plasmas. As one-dimensional plasmas, the system composed of charged sheets has been extensively investigated.[9] The potential in this case is proportional to mutual distances. In our one-dimensional plasmas, particles interact through potential proportional to the inverse of the mutual distance. We have this kind of one-dimensional plasmas in the central part of Penning (and Paul) traps with cylindrical symmetry, when the effect of confining force is strong. We also note that in quantum wires in semiconductors we have a system of classical charges with one degree of freedom interacting via the inverse r potential under appropriate conditions.

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II. CHARACTERISTIC PARAMETERS

A. Ions in Traps

Ions in Penning (and Paul) traps have a variety of ordered structures at low temperatures.[3, 4, 5, 6, 7] These structures are determined as a result of the competition between the confining force and mutual Coulomb repulsion.[7] When the former is sufficiently strong, ions are aligned on a line, and with its decrease, we have one-dimensional lattices on a cylinder, two concentric cylinders (layers), three concentric cylinders, and so on.

We first consider the ions on a line: The cases of ions on cylindrical surfaces will be discussed in the last section. As characteristic energies of this system, we have, in addition to the thermal energy k_BT, the Coulomb interaction at mean distance E_C and the Fermi energy E_F given by

$$E_C = nq^2, \qquad E_F = (\pi^2/8)(\hbar^2 n^2/m). \tag{1}$$

By q, n, T, and m, we denote the charge, the line density, the temperature, and the ionic mass, respectively. (The energy E_F characterizes the energy for which we have to take the effect of quantum statistics into account even if the ion is a boson.) These parameters are compared for Be_9^+ ion in Fig.1.

We observe that these plasmas are usually in the classical domain where $k_BT > E_F$ due to large ionic mass. We there define the Coulomb coupling parameter γ by

$$\gamma = E_C/k_BT = nq^2/k_BT. \tag{2}$$

As is shown in Fig.1, we can be in either the weakly coupled domain $\gamma < 1$ (A) or the strongly coupled domain $\gamma > 1$ (B) according to experimental conditions.

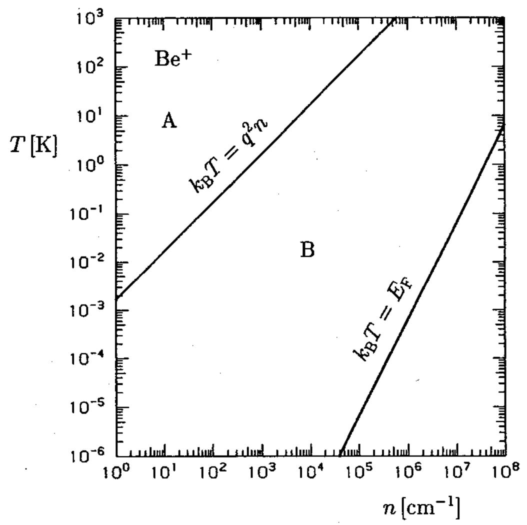

FIGURE 1. Characteristic parameters of ions in traps.

B. Ions in Storage Rings

In ion storage rings, alternating quadrupole magnetic fields serve as confining force which focuses ions in directions perpendicular to the beam. Sufficiently low parallel temperatures of the order of meV are attained by employing the electron and laser coolings, while the perpendicular temperatures are much higher and of the order of eV.

Ions in storage rings can be regarded as classical one-dimensional plasmas along the direction of the beam.[10] The effective interactions in this system are modified by the finite perpendicular extensions and the existence of grounded wall surrounding the beam. Correlations between ions can be observed in the spectrum of Schotkky noise picked up by electrodes placed near the beam: In the domains of strong coupling we have a reduction of integrated spectrum and the appearance of collective mode.[11] These effects can be accounted for by regarding the ion beam as one-dimensional plasma with an effective interaction.[10]

C. Electrons in Quantum Wires

In microstructures such as quantum wires in semiconductors, electrons (and holes) are confined by the potential due to the band offset between different semiconductors. Typical potential depth in structures composed of GaAs and $Al_xGa_{1-x}As$ is of the order of 100meV. For electrons in quantum wires, the mass is to be replaced by the effective mass and the characteristic Coulomb energy is screened by the dielectric constant ε as $E_C = q^2n/\varepsilon$. (We assume that the difference in energies of the ground and the first excited states in the plane perpendicular to the wire ΔE satisfies $\Delta E > Max(k_BT, E_F)$, and only the ground state is occupied.)

In the classical domain where $k_BT > E_F$, the parameter $\gamma = E_C/k_BT = q^2n/\varepsilon k_BT$ characterizes the importance of Coulomb interactions;

$$\gamma < 1 \quad \text{weakly coupled (A)}, \qquad \gamma > 1 \quad \text{strongly coupled (B)}. \tag{3}$$

In the degenerate case $k_BT < E_F$, the ratio $E_C/E_F = (8/\pi^2)R_s$ characterizes the importance of Coulomb interactions;

$$R_s < 1 \quad \text{weakly coupled (C)}, \qquad R_s > 1 \quad \text{strongly coupled (D)}. \tag{4}$$

These three energies are compared with one another in Fig.2 for electrons in quantum wires composed of GaAs with $m = 0.067m_e$ and $\varepsilon = 13.1$, m_e being the electronic mass. In the domain A, our system is classical and weakly coupled, in B, classical and strongly coupled, in C, degenerate and weakly coupled, and in D, degenerate and strongly coupled.

We note that, except for very low temperatures and high densities, electrons in quantum wires are classical with respect to the motion along the wire. In other words, we have one-dimensional classical plasmas in quantum wires under a wide range of conditions.

We keep the dielectric constant ε in following expressions: $\varepsilon = 1$ for ions in traps and storage rings.

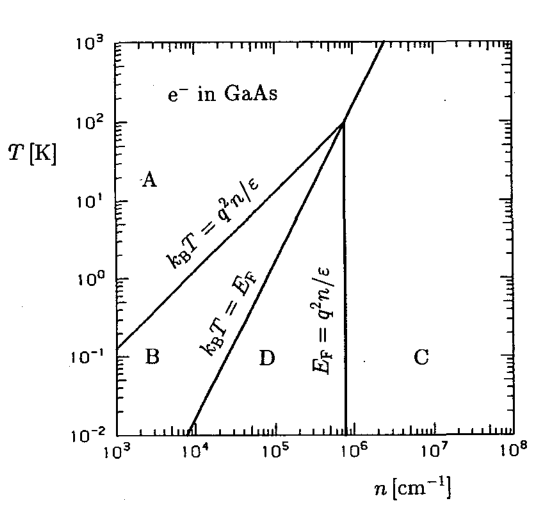

FIGURE 2. Characteristic parameters of electrons in quantum wires.

III. RANDOM PHASE APPROXIMATION

In the domain of weak coupling, we may apply the random phase approximation to our one-dimensional plasmas. When we cutoff the interaction $q^2/\varepsilon|z|$ at δ as $q^2/\sqrt{z^2+\delta^2}$, its Fourier component is given by $2K_0(k\delta) \sim 2q^2(-\ln k)$ ($K_0(x)$ is the modified Bessel function) and the plasmon dispersion $\omega(k)$ and the static form factor $S(k)$ are calculated as

$$\omega^2(k) \sim \frac{2q^2n}{\varepsilon m}k^2(-\ln k), \tag{5}$$

$$S(k) \sim \left(1+\frac{2nq^2}{\varepsilon k_B T}(-\ln k)\right)^{-1}, \tag{6}$$

respectively.[10]

IV. STRONGLY COUPLED DOMAIN

The Hamiltonian of our system may be written as

$$H=\sum_i \frac{p_i^2}{2m}+\frac{q^2}{2\varepsilon}\sum_{i\neq j}\frac{1}{|z_i-z_j|}+H_b. \tag{7}$$

Here z_i denotes the position of i-th electron and H_b, the effect of neutralizing background. Let us note that, since our system is one-dimensional and composed of the same particles, we can always number our particles in the order of their distance from some arbitrary taken origin. The inverse r potential also assures that this ordering is maintained throughout the time development of our system.[12] We thus have

$$z_i \geq z_{i-1}. \tag{8}$$

When the couplings between classical charges are sufficiently strong, it is natural to expect those charges have correlations which resemble those in a solid phase. For our system in the strongly coupled regime, we therefore imagine a lattice of particles with positions $\{z_i^0\}$ and express the positions of particles $\{z_i\}$ by deviations from those in the lattice $\{\zeta_i\}$ as

$$z_i = z_i^0 + \zeta_i \tag{9}$$

with

$$z_i^0 = \frac{1}{n}i. \tag{10}$$

It is well known that we do not have solids in one dimension due to enhanced effect of thermal fluctuations: The mean square displacement calculated in the harmonic approximation diverges and the Debye-Waller factor reduces to zero due to the contribution of acoustic modes with long wavelengths.[13] In this respect, we emphasize that, in our calculation, it is *not* necessary to assume the existence of such a lattice.

In terms of ζ_i, $|z_i-z_j| = |(i-j)/n+(\zeta_i-\zeta_j)|$, and the Hamiltonian is rewritten as

$$H=\sum_i \frac{p_i^2}{2m}+\frac{q^2}{2\varepsilon}\sum_{i\neq j}\frac{1}{|(i-j)/n+(\zeta_i-\zeta_j)|}+H_b, \tag{11}$$

where $p_i = m(d\zeta_i/dt)$ conjugate to ζ_i. We now expand each term in the potential with respect to $\zeta_i-\zeta_j$. Within the harmonic approximation, we have

$$H=\sum_i \frac{p_i^2}{2m}+\frac{q^2n}{2\varepsilon}\sum_{i\neq j}\frac{1}{|i-j|}+H_b+\frac{q^2n^3}{2\varepsilon}\sum_{i\neq j}\frac{1}{|i-j|^3}(\zeta_i-\zeta_j)^2. \tag{12}$$

The second and third terms on the right hand side give the Madelung energy of linear chain of charges. When the effect of neutralizing backgroung charge is properly taken into account, they reduce to

$$U_M/2 + \ln(Rn), \tag{13}$$

where $U_M = -0.231803\cdots$ and R is the radius of cylinder containing neutralizing background.[14]

The rest of the Hamiltonian (12) gives a collection of harmonic oscillations. Their dispersion relation is calculated as

$$\omega^2(k) = \frac{4q^2n^3}{\varepsilon m} F\left(\frac{k}{n}\right) \qquad |k| < \frac{\pi}{n}, \tag{14}$$

where

$$F(x) = \sum_{j=1}^{\infty} \frac{1}{j^3}(1 - \cos jx). \tag{15}$$

For long wavelengths, (14) reduces to

$$\omega^2(k) \sim \frac{2q^2n}{\varepsilon m} k^2(-\ln k). \tag{16}$$

We show the dispersion relation in Fig.3.

To confirm the applicability of the harmonic approximation, we now calculate $<(\zeta_i - \zeta_j)^2>$. Here $<>$ denotes the thermal average. At the temperature T, we have

$$<(\zeta_i - \zeta_j)^2> n^2 = \frac{1}{\pi\gamma}\int_0^\pi dx \frac{\sin^2\left(\frac{(i-j)}{2}x\right)}{F(x)} \tag{17}$$

in the harmonic approximation. We plot the ratio of $<(\zeta_i - \zeta_j)^2>$ to $(i-j)^2/n^2$ normalized by $1/\gamma$ in Fig.4. We note that the average $<(\zeta_i - \zeta_j)^2>$ is finite even if the mean square displacement $<\zeta_i^2>$ diverges and the ratio plotted in Fig.4 is a decreasing function of $|i-j|$ with the maximum of $0.4/\gamma$ at $|i-j| = 1$. When $|i-j| \gg 1$, we have

$$\frac{<(\zeta_i - \zeta_j)^2>}{(i-j)^2/n^2} \sim \frac{1}{2\gamma}\frac{1}{|i-j|\ln|i-j|}. \tag{18}$$

We also note that this asymptotic expression works as a good approximation for $|i-j| \geq 2$.

The approximation we applied to the Hamiltonian (12) is the expansion of $1/|z_i - z_j| = 1/|(i-j)/n + (\zeta_i - \zeta_j)|$ with respect to $\zeta_i - \zeta_j$. Our harmonic approximation is thus applicable at least as a first approximation when $\gamma > 1$ or in the domain of strong coupling.

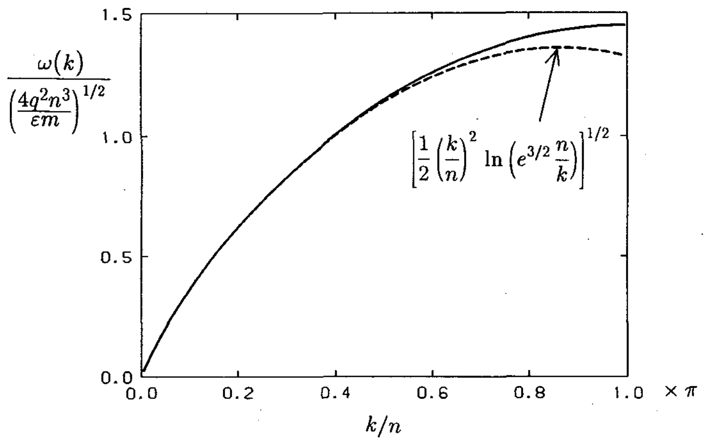

FIGURE 3. Dispersion relation of one-dimensional plasmon.

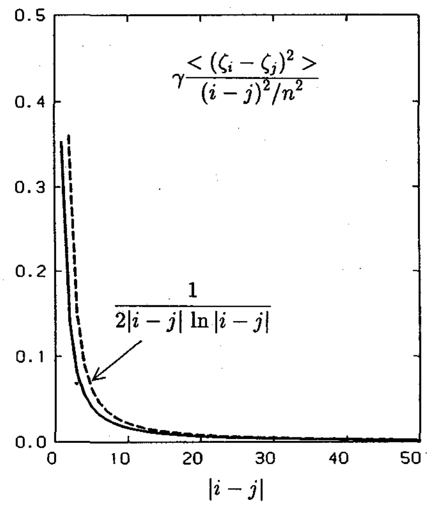

FIGURE 4. Relative displacement from (fictitious) lattice points.

The static correlation between charges is characterized by the structure factor $S(k)$ and the pair distribution function $g(z)$ given by

$$S(k) = \frac{1}{N} < \rho_k \rho_{-k} >, \tag{19}$$

$$g(z) - 1 = \frac{n}{2\pi} \int_{-\infty}^{\infty} dk \exp(ikz)[S(k) - 1]. \tag{20}$$

Here $\rho_k = \sum_j \exp(-ikz_j)$ is the density fluctuation and N is the total number of charges. Within the harmonic approximation, the structure factor and the pair distribution function are calculated as

$$S(k) = 1 + 2\sum_{j=1}^{\infty} \cos\left(j\frac{k}{n}\right) \exp\left[-\frac{k^2}{2} < (\zeta_0 - \zeta_j)^2 >\right]$$
$$= 1 + 2\sum_{j=1}^{\infty} \cos\left(j\frac{k}{n}\right) \exp\left[-\frac{1}{4\pi}\frac{\varepsilon k_B T}{q^2 n}\frac{k^2}{n^2}\int_0^{\pi} dx \frac{1}{F(x)}[1 - \cos(jx)]\right], \tag{21}$$

$$g(z) = \frac{1}{(2\pi)^{1/2}} \sum_{j=1}^{\infty} \frac{1}{n < (\zeta_0 - \zeta_j)^2 >^{1/2}}$$
$$\times \left\{\exp\left[-\frac{(z - j/n)^2}{2 < (\zeta_0 - \zeta_j)^2 >}\right] + \exp\left[-\frac{(z + j/n)^2}{2 < (\zeta_0 - \zeta_j)^2 >}\right]\right\}. \tag{22}$$

We show the structure factor and the correlation function in Fig.5 for several values of coupling constant γ.

The dynamic structure factor $S(k, \omega)$ is the Fourier transform of the intermediate scattering function $< \rho_k(t)\rho_{-k}(t = 0) > /N$ where $\rho_k(t) = \sum_j \exp[-ikz_j(t)]$:

$$S(k, \omega) = \frac{1}{N}\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \exp(i\omega t) < \rho_k(t)\rho_{-k}(t = 0) > . \tag{23}$$

The intermediate scattering function is calculated as

$$\frac{1}{N} < \rho_k(t)\rho_{-k}(t = 0) >= \sum_{j=-\infty}^{\infty} \exp\left(-j\frac{k}{n}\right) \exp\left[-\frac{k^2}{2} < \{\zeta_0(t = 0) - \zeta_j(t)\}^2 >\right]$$
$$= \sum_{j=-\infty}^{\infty} \exp\left(-j\frac{k}{n}\right) \exp\left[-\frac{1}{4\pi}\frac{\varepsilon k_B T}{q^2 n}\frac{k^2}{n^2}\int_0^{\pi} dx \frac{1}{F(x)}[1 - \cos\{jx - \omega(nx)t\}]\right]. \tag{24}$$

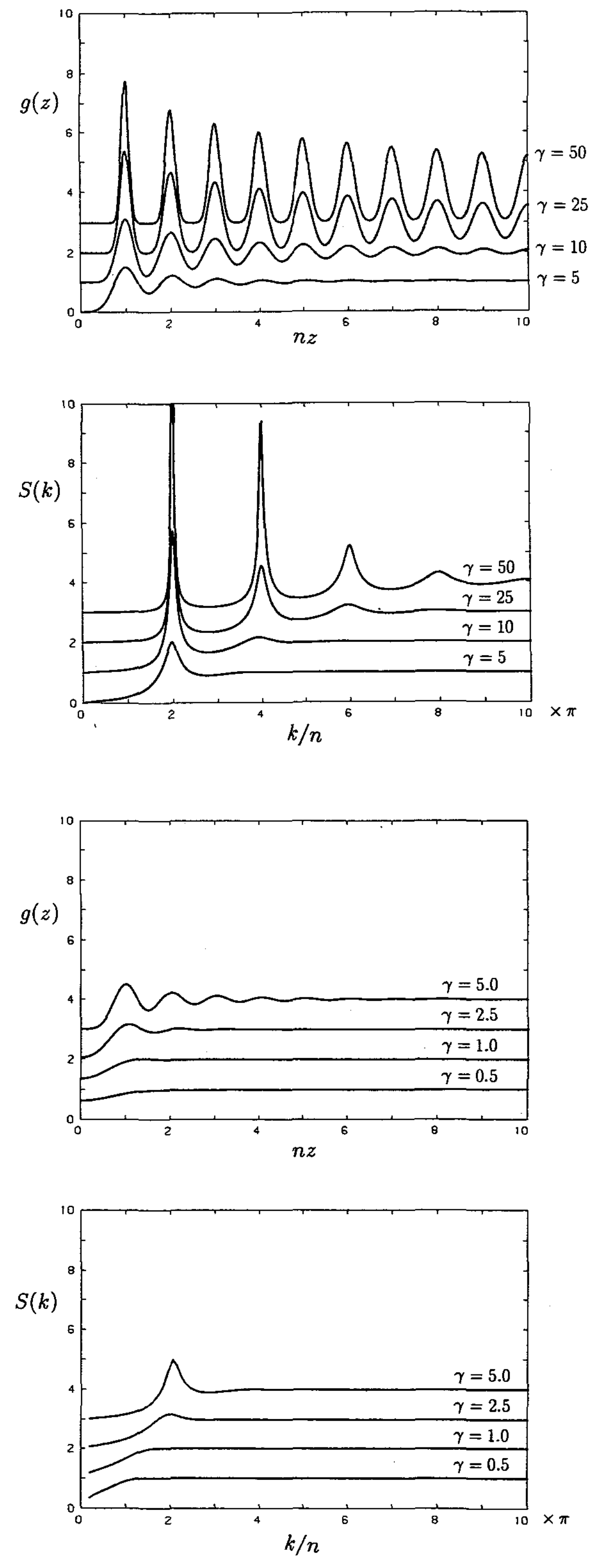

FIGURE 5. Pair distribution function and static form factor.

V. ONE-DIMENSIONAL LATTICE ON CYLINDERS

We here consider one-dimensional lattices of charges on a cylinder which are realized in Penning and Paul traps.[7, 15] We write the position of i-th charge $\mathbf{r}_i$ referring to the lattice (denoted by the superscript 0) as

$$\mathbf{r}_i = (\mathbf{R}_i^0 + \mathbf{R}_i, z_i^0 + \zeta_i). \tag{25}$$

Here $\mathbf{R}_i^0$ and $\mathbf{R}_i$ have x and y components. Expanding the mutual interaction $q^2/|\mathbf{r}_i - \mathbf{r}_j|$ with respect to $\zeta_{ij} = \zeta_i - \zeta_j$ and $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$, we have the deviation of the interaction potential from the Madelung energy to the second order as

$$-\frac{1}{2}\frac{q^2}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij}^0)^2]^{3/2}}(\zeta_{ij}^2 + \mathbf{R}_{ij}^2) + \frac{3}{2}\frac{q^2}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij}^0)^2]^{5/2}}(z_{ij}^0\zeta_{ij} + \mathbf{R}_{ij}^0 \cdot \mathbf{R}_{ij})^2, \tag{26}$$

where $z_{ij}^0 = z_i^0 - z_j^0$ and $\mathbf{R}_{ij}^0 = \mathbf{R}_i^0 - \mathbf{R}_j^0$. The equations of motion for ζ_i and $\mathbf{R}_i$ are given by

$$m\frac{d^2\zeta_i}{dt^2} = 2\sum_{j(\neq i)} \frac{2(z_{ij}^0)^2 + (\mathbf{R}_{ij})^2}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij})^2]^{5/2}}\zeta_{ij} + 6\sum_{j(\neq i)} \frac{z_{ij}^0}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij})^2]^{5/2}}\mathbf{R}_{ij}^0 \cdot \mathbf{R}_{ij}, \tag{27}$$

$$m\frac{d^2\mathbf{R}_i}{dt^2} = -k\mathbf{R}_i + 2\sum_{j(\neq i)} \frac{1}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij})^2]^{5/2}}[((z_{ij}^0)^2 + (\mathbf{R}_{ij})^2)\mathbf{R}_{ij} - 3\mathbf{R}_{ij}^0\mathbf{R}_{ij}^0 \cdot \mathbf{R}_{ij}]$$
$$+ 6\sum_{j(\neq i)} \frac{z_{ij}^0}{[(z_{ij}^0)^2 + (\mathbf{R}_{ij})^2]^{5/2}}\mathbf{R}_{ij}^0\zeta_{ij}, \tag{28}$$

respectively. Here k in the equation for $\mathbf{R}_i$ is the confining force constant. When we note that $|\mathbf{R}_{ij}^0| \le$ (diameter of cylinder) while $|z_{ij}^0| \propto |i-j|$ and the convergence of summations are relatively slow, the equations of motion for ζ_i and $\mathbf{R}_i$ are approximately decoupled as

$$m\frac{d^2}{dt^2}\zeta_i \sim 4\sum_{j(\neq i)} \frac{\zeta_{ij}}{|z_{ij}^0|^3}, \tag{29}$$

$$m\frac{d^2}{dt^2}\mathbf{R}_i \sim -k\mathbf{R}_i + 2\sum_{j(\neq i)} \frac{1}{|z_{ij}^0|^3}\mathbf{R}_{ij}. \tag{30}$$

The longitudinal fluctuations in these lattices are thus given by the dispersion relation (14) and density fluctuation is given by (21).

VI. CONCLUSION

We have shown that the strongly coupled classical one-dimensional plasma with $1/r$ interaction can be described by the harmonic approximation. The harmonic approximation which is originally devised for crystalline state does apply to one-dimensional system where crystalline order cannot exist due to enhanced effect of long wavelength fluctuations. These results apply to one-dimensional string of charges in long traps and also to electrons in quantum wires under appropriate conditions. Dynamical simulations of this system are in progress with preliminary results confirming our calculations. Their full account will be given elsewhere.

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