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On Resonance in Periodically Forced Oscillators and Coupled Systems of Excitable Systems and Nonlinear Oscillators

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Abstract

We analyze some mathematical problems that arise in studies of phenomena observed in the cardiac action. We illustrate a method to characterize the response of a nonlinear oscillator to an external forcing, and introduce some numerical results. We also introduce some results of numerical computation in an example of a coupled system of an excitable system and a nonlinear oscillator.

KEYWORDS: periodic forcing, nonlinear oscillators, excitable systems, couples systems.

1. Introduction

Electrical oscillation is one of features that cardiac tissue cells possess. Another important factor in the mechanism of the mammalian heart is synchronization of the oscillatory outputs in a population of cells. Some of mathematical problems relevant to such phenomena are formulated in the framework of coupled oscillators that model the dynamics in the interaction of oscillatory units [2], [6]. The results of these studies include theories concerning synchronization or phase-rocking within a population of oscillatory units. However, it is also important to understand how the population respond to an external forcing. Here, we consider a mathematical problem to study the response of an oscillatory unit to an external forcing.

We suppose that a function $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is k-times continuously differentiable and that $k \geq 2$. We also suppose that the system of ordinary differential equations

$$\frac{dx}{dt} = f(x) \tag{1}$$

has a nonconstant periodic solution with $x = \eta(t)$ with least period $T_0 > 0$. We first study the perturbed system of (1):

$$\frac{dx}{dt} = f(x) + \delta\psi(t), \tag{2}$$

where the function $\psi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is periodic with period T_1. In Section 2, we describe a method to use the transformation $x = \eta(\theta) + \Phi(\theta)y$, and to obtain the system of ordinary differential equations in which θ and y are the dependent variables. We call these equations the phase-amplitude equations. Here, the columns of the $n \times (n-1)$ matrix $\Phi(\theta)$ constitute an orthonormal basis of the orthogonal complement of $\eta'(\theta) = f(\eta(\theta))$. Using the phase-amplitude equations, we introduce a method to characterize the response of the system to the forcing exerted by $\psi(t)$. In Section 3, we consider (2) with a particular planar oscillator equivalent to the Bonhoeffer-van der Pol equations [4], and with a piecewise constant forcing function. We introduce a numerical result obtained by using the phase-amplitude equations.

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We also study the following class of systems of ordinary differential equations.

$$\begin{aligned}\frac{dx_1}{dt} &= f(x_1) + \delta_1 h_1(x_1, x_2, \delta), \\ \frac{dx_2}{dt} &= g(x_2) + \delta_2 h_2(x_1, x_2, \delta).\end{aligned} \tag{3}$$

Here, $g : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, $h_1 : \mathbf{R}^{m+n+1} \longrightarrow \mathbf{R}^n$, and $h_2 : \mathbf{R}^{m+n+1} \longrightarrow \mathbf{R}^m$. We assume that the system of ordinary differential equation

$$\frac{dx}{dt} = g(x) \tag{4}$$

is excitable. The dynamics of such a system often models response of a muscular tissue to electrical stimuli, called excitation. Studies of periodically forced excitable systems are found in [1] and [7]. Here, we introduce a numerical solution of (3) in Section 4, in which the systems at (1) and (4) are given by Bonhoeffer-van der Pol equations, and show that an excitable system can become oscillatory when coupled with an oscillator.

2. Characterization of the perturbed system

We recall $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is k-times continuously differentiable with $k \geq 2$, and that the system at (1) has a nonconstant periodic solution $x = \eta(t)$ with least period $T_0 > 0$. Then there is an $n \times (n-1)$-matrix $\Phi(\theta)$ whose entries are all k-times continuously differentiable functions of θ, such that

$$\begin{aligned}\Phi(\theta + T_0) &= \Phi(\theta), \\ \Phi(\theta)^T \Phi(\theta) &= I, \\ \Phi(\theta)^T f(\eta(\theta)) &= O,\end{aligned}$$

for all $\theta \in \mathbf{R}$. Here I is the $(n-1) \times (n-1)$-identity matrix, and O is the $(n-1)$-dimensional zero vector. We define the transformation between a neighborhood of the orbit of the periodic solution and $\mathbf{R} \times U$, where U is a neighborhood of the origin in $\mathbf{R}^{n-1}$, by

$$x = \eta(\theta) + \Phi(\theta) y.$$

Then we obtain the following system of ordinary differential equations for θ and y:

$$\begin{aligned}\frac{d\theta}{dt} &= 1 + \Theta(t, \theta, y, \delta), \\ \frac{dy}{dt} &= B(\theta) y + Y(t, \theta, y, \delta),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\Theta(t,\theta,y,\delta) &= \frac{f(\eta(\theta))^T \left[f(\eta(\theta) + \Phi(\theta)y) - f(\eta(\theta)) - \Phi'(\theta)y + \delta\psi(t)\right]}{f(\eta(\theta))^T \left[f(\eta(\theta)) + \Phi'(\theta)y\right]}, \\ B(\theta) &= \Phi(\theta)^T \left[Df(\eta(\theta))\Phi(\theta) - \Phi'(\theta)\right], \\ Y(t,\theta,y,\delta) &= \Phi(\theta)^T \left[f(\eta(\theta) + \Phi(\theta)y) - f(\eta(\theta)) - Df(\eta(\theta))\Phi(\theta)y + \delta\psi(t) - \Theta(t,\theta,y,\delta)\Phi'(\theta)y\right].\end{aligned}$$

We note that the functions $\Theta(t, \theta, y, \delta)$ and $Y(t, \theta, y, \delta)$ are periodic in t with period T_1, and that they are also periodic in θ with period T_0. It can be shown that

$$\begin{aligned}\Theta(t,\theta,y,\delta) &\sim \mathcal{O}(\|y\| + |\delta|), \\ Y(t,\theta,y,\delta) &\sim \mathcal{O}(\|y\|^2 + |\delta|).\end{aligned}$$

Consider the variational system of (1) with respect to the periodic solution $x = \eta(t)$:

$$\frac{dx}{dt} = A(t)x, \tag{6}$$

where $A(t) = Df(\eta(t))$. 1 is a multiplier of the linear system at (6). We denote by the remaining $n-1$ multipliers by $\lambda_2, \dots \lambda_n$. It can be shown that $\lambda_2, \dots \lambda_n$ are the characteristic multipliers of the following linear system [8]:

$$\frac{dy}{dt} = B(t)y.$$

Suppose that $(\theta, y) = (\theta(t, \theta_0, y_0, \delta), y(t, \theta_0, y_0, \delta))$ is the solution of (5) with the initial value

$$(\theta(0, \theta_0, y_0, \delta), y(0, \theta_0, y_0, \delta)) = (\theta_0, y_0).$$

Set $x_0 = \eta(\theta_0) + \Phi(\theta_0)y_0$. Then $x = x(t, x_0, \delta) = \eta(\theta(t, \theta_0, y_0, \delta)) + \Phi(\theta(t, \theta_0, y_0, \delta))y(t, \theta_0, y_0, \delta)$ is the solution of (2) with the initial value $x(0, x_0.\delta) = x_0$. In particular, if there are positive integers i and j such that

$$\begin{aligned} \theta(jT_1, \theta_0, y_0, \delta) &= \theta_0 + iT_0, \\ y(jT_1, \theta_0, y_0, \delta) &= y_0, \end{aligned}$$

then $x(t, x_0, \delta)$ is a periodic solution with period jT_1. We define the rotation number $\rho(\theta_0, y_0, \delta)$ by

$$\rho(\theta_0, y_0, \delta) = \frac{1}{T_0} \lim_{k \to \infty} \frac{\theta(kT_1, \theta_0, y_0, \delta) - \theta_0}{k}.$$

If the rotation number is a rational number,

$$\rho(\theta_0, y_0, \delta) = \frac{i}{j},$$

for all $\theta_0 \in \mathbf{R}$ and for all y_0 in some neighborhood of the origin, then we say that the system at (2) is in $j : i$ resonance. When $|\lambda_i| < 1$, $i = 2, 3, \dots, n$, (2) has a one-parameter family of invariant manifolds $x = X(t, \theta, \delta)$ that exists for all small $|\delta|$, such that $X(t+T_1, \theta, \delta) = X(t, \theta, \delta)$ and $X(t, \theta+T_0, \delta) = X(t, \theta, \delta)$ for all t and θ [5]. These invariant manifolds lie in a neighborhood of the cylinder $\Gamma \times \mathbf{R}$, where Γ is the orbit of $\eta(t)$. Then, (5) has a one-parameter family of integral manifolds $y = Y(t, \theta, \delta)$ that exists for all small $|\delta|$, such that $Y(t + T_1, \theta, \delta) = Y(t, \theta, \delta)$ and $Y(t, \theta + T_0, \delta) = Y(t, \theta, \delta)$ for all t and θ. Then for all sufficiently small $|y_0|$, the rotation number coincide with the rotation number for the ordinary differential equation

$$\frac{d\psi}{d\tau} = \frac{T_1}{T_0} [1 + \Theta(T_1\tau, T_0\psi, Y(T_1\tau, T_0\psi, \delta), \delta)]$$

defined in [3].

3. A numerical solution of the phase-amplitude equations

In this section, we consider the following system of ordinary differential equations

$$\begin{aligned} \frac{dv}{dt} &= \frac{1}{\epsilon}\left(v - \frac{v^3}{3} - w + a\right), \\ \frac{dw}{dt} &= v - bw + c, \end{aligned} \tag{7}$$

where a, b, c, and ϵ are constants. This system is equivalent to the Bonhoeffer-van der Pol equations [4]. For some appropriate values of parameters, the system has a nonconstant periodic solution. An example of such a solution is numerically generated with the following values of the parameters.

$$\begin{aligned} a &= 0, \\ b &= 0.4, \\ c &= 0, \\ \epsilon &= 0.1. \end{aligned} \tag{8}$$

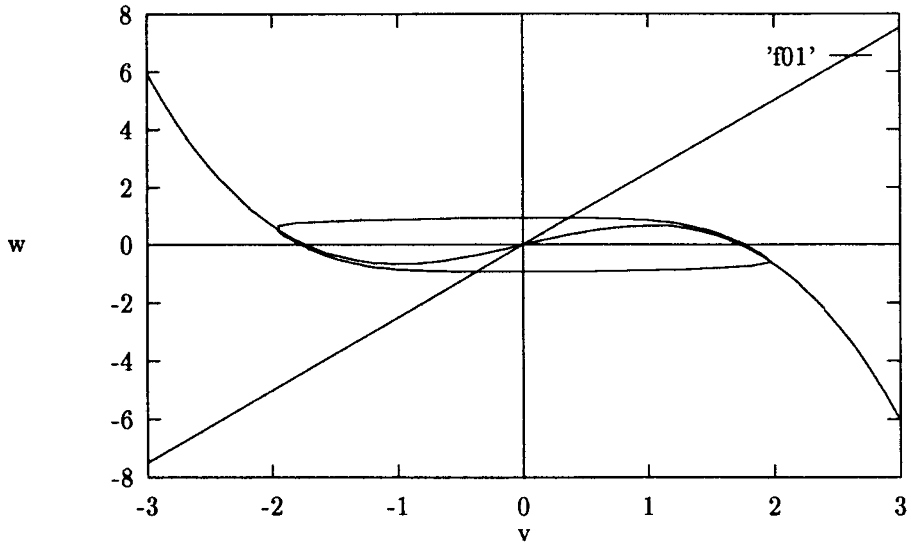

Figure 1: The orbit of the periodic solution of (7). A numerically constructed orbit of a periodic solution of (7) with the values of the parameters at (8) is shown in the (v, w)-plane. The nullclines $w = v - v^3 + a$ and $w = (v + c)/b$ are also shown. Here, the period T_0 of the periodic solution is given approximately by $T_0 \approx 2.896420$.

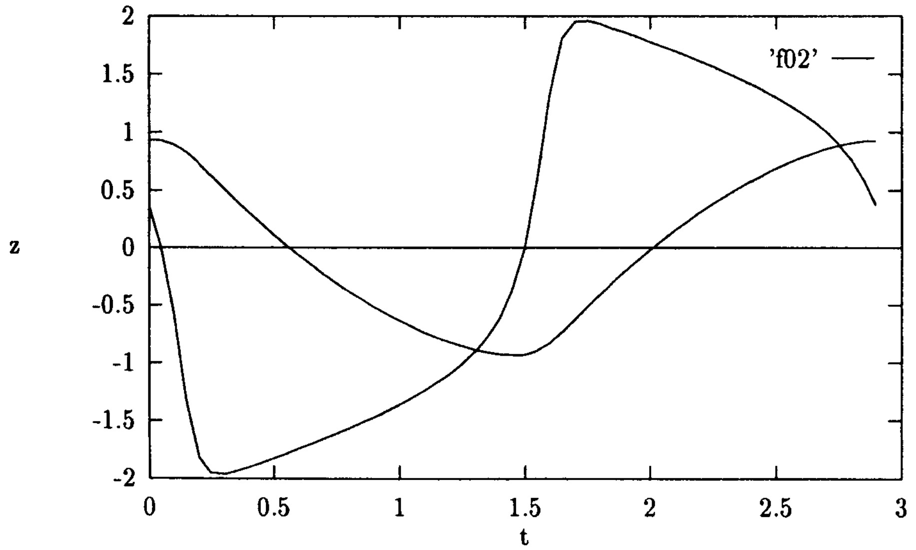

Figure 2: The components of the periodic solution of (7). The graphs $z = v(t)$ and $z = w(t)$ are shown in the (t, z)-plane. Here, $v = v(t)$ and $w = w(t)$ are the components of the numerical solution whose orbit is shown in Figure 1. $v(0)$ and $w(0)$ are given approximately by $v(0) \approx 0.373396$ and $w(0) \approx 0.933490$.

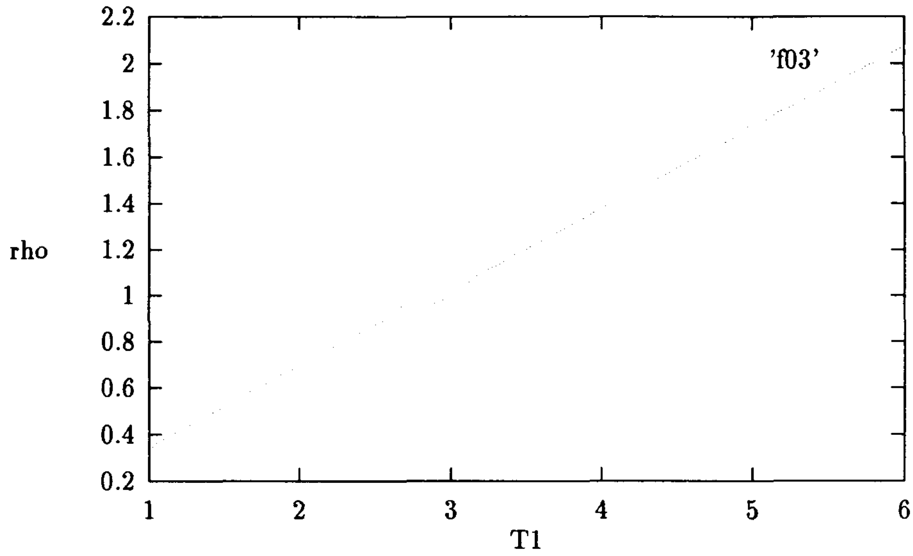

Figure 3: The rotation number for $\delta = 1$. A numerically computed rotation number for $0.1 \le T_1 \le 6$ is shown. Here, $\delta = 1$, and the other parameters are as set at (8) and (10).

Figure 1 shows the orbit of the periodic solution and Figure 2 shows its components. We also consider the piecewise constant forcing function, whose components are defined by

$$\psi_1(t) = \begin{cases} A_0, & kT_1 \le t < (k+\sigma)T_1, \\ A_1, & (k+\sigma)T_1 \le t < (k+1)T_1, \quad k = 0, \pm 1, \pm 2, \ldots, \quad 0 < \sigma < 1, \end{cases}$$
$$\psi_2(t) = 0,$$

where A_0 and A_1 are constants. When these functions are used, the system at (2) can be written in the following component form.

$$\begin{aligned} \frac{dv}{dt} &= \frac{1}{\epsilon}\left(v - \frac{v^3}{3} - w + a\right) + \delta\psi_1(t), \\ \frac{dw}{dt} &= v - bw + c. \end{aligned} \tag{9}$$

The phase-amplitude equations illustrated in Section 2 are numerically analyzed with the values of the parameters set at (8), and with the following values of the parameters A_0, A_1, and σ.

$$\begin{aligned} A_0 &= 0, \\ A_1 &= 1, \\ \sigma &= 0.5. \end{aligned} \tag{10}$$

Figure 3 and 4 shows the dependence of the rotation number ρ on the forcing period T_1. Figure 5 shows the trajectory of a solution of (9), and Figure 6 shows its components.

4. A numerical solution of a coupled system

In this section, we introduce a numerical solution of a coupled system of an excitable system and a nonlinear oscillator. We consider two class of the Bonhoeffer-van der Pol equations.

$$\begin{aligned} \frac{dv_i}{dt} &= \frac{1}{\epsilon_i}\left(v_i - \frac{v_i^3}{3} - w_i + a_i\right), \\ \frac{dw_i}{dt} &= v_i - b_i w_i + c_i, \qquad i = 1, 2, \end{aligned} \tag{11}$$

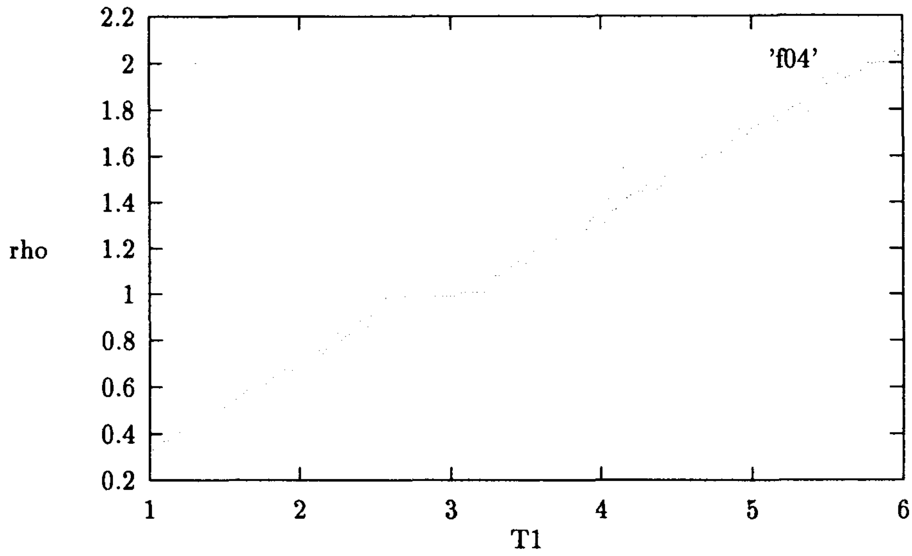

Figure 4: The rotation number for $\delta = 2$. A numerically computed rotation number for $0.1 \leq T_1 \leq 6$ is shown. Here, $\delta = 2$, and the other parameters are as set in the computation for Figure 3.

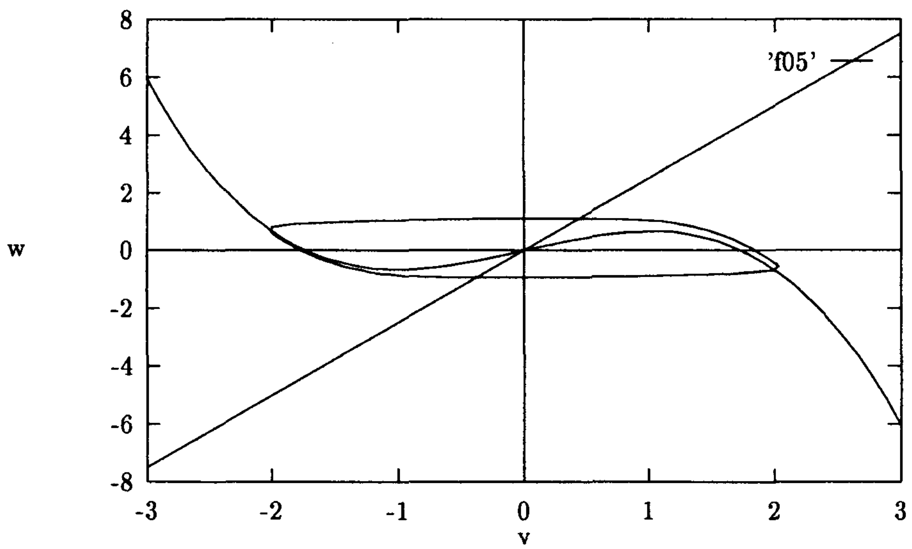

Figure 5: The trajectory of a solution of (9). A numerically constructed periodic solution of (9) for one forcing cycle is shown in the (v, w)-plane. The nullclines of (7), $w = v - v^3 + a$ and $w = (v + c)/b$, are also shown. Here, $\delta = 2$ and $T_1 = 3$. The other parameters are as set in the computation for Figure 4.

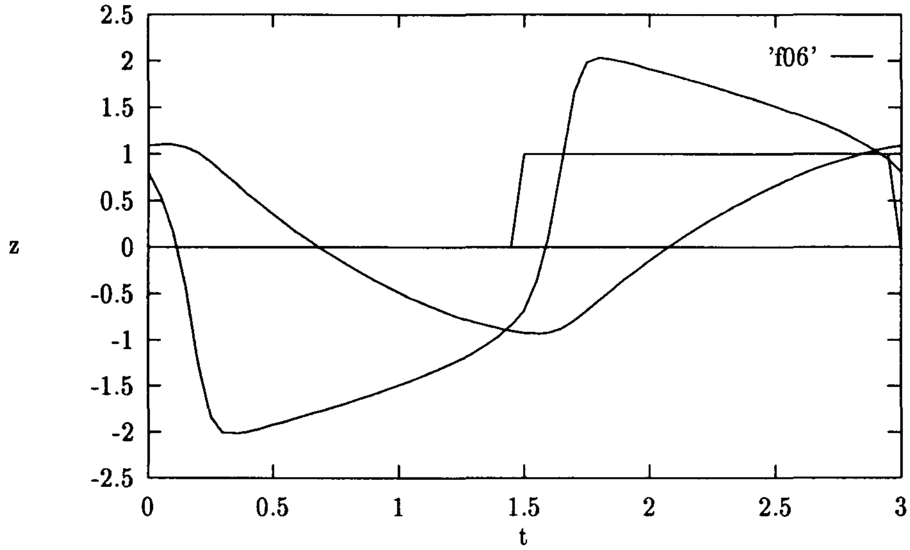

Figure 6: The solution of (9). The graphs $z = v(t)$ and $z = w(t)$ are shown in the (t, z)-plane. Here, $v = v(t)$ and $w = w(t)$ are the components of the numerical solution whose trajectory is shown in Figure 5. $v(0)$ and $w(0)$ are given approximately by $v(0) \approx 0.811471$ and $w(0) \approx 1.087351$. The graph $z = \psi_1(t)$ is also shown.

where a_i, b_i, c_i, and ϵ_i are constants. We set the following values of the parameters a_1, b_1, c_1, and ϵ_1.

$$\begin{aligned} a_1 &= 0, \\ b_1 &= 0.4, \\ c_1 &= 0, \\ \epsilon_1 &= 0.1. \end{aligned} \tag{12}$$

Then the system at (11) has a nonconstant periodic soulution for $i = 1$ as is shown in Section 3. We also set the following values of the parameters a_2, b_2, c_2, and ϵ_2.

$$\begin{aligned} a_2 &= -3, \\ b_2 &= 0.4, \\ c_2 &= 0, \\ \epsilon_2 &= 0.05. \end{aligned} \tag{13}$$

Then it can be shown that the system at (11) is excitable for $i = 2$.

We solve the following coupled system numerically.

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{\epsilon_1}\left(v_1 - \frac{v_1^3}{3} - w_1 + a_1\right) + \delta_1(v_2 - v_1), \\ \frac{dw_1}{dt} &= v_1 - b_1 w_1 + c_1, \\ \frac{dv_2}{dt} &= \frac{1}{\epsilon_2}\left(v_2 - \frac{v_2^3}{3} - w_2 + a_2\right) + \delta_2(v_1 - v_2), \\ \frac{dw_2}{dt} &= v_2 - b_2 w_2 + c_2. \end{aligned} \tag{14}$$

In addition to the values of the parameters set at (12) and (13), we set the following values of the remaining parametrs.

$$\begin{aligned} \delta_1 &= 1, \\ \delta_2 &= 10. \end{aligned} \tag{15}$$

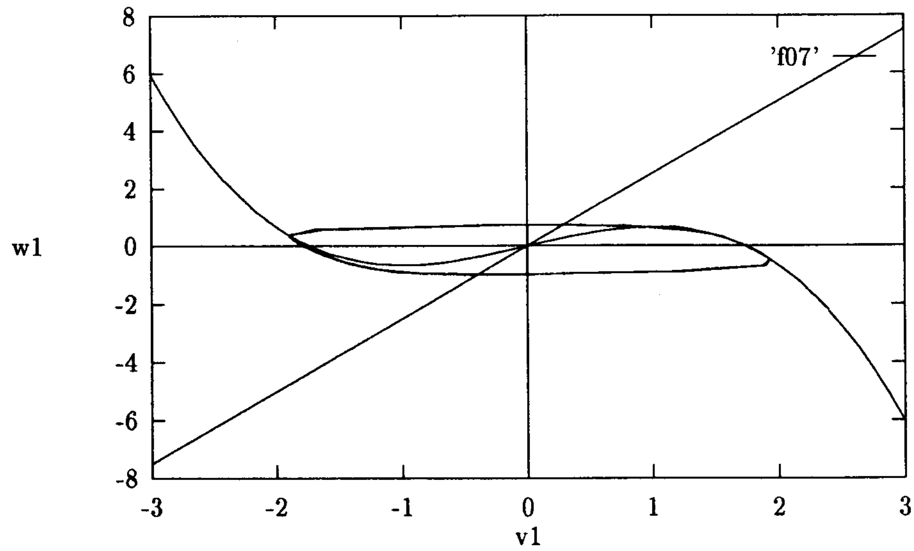

Figure 7: The trajectory of a solution of the coupled system. The (v_1, w_1)-components of the trajectory of a numerical solution of the system at (14) with the values of the parameters given at (12), (13), and (15) is shown in the (v_1, w_1)-plane. The nulclines $w_1 = v_1 - v_1^3/3 + a_1$ and $w_1 = (v_1 + c_1)/b_1$ are also shown.

A numerical solution of the coupled system at (14) is illustrated in Figures 7 - 10. In this example, we note that the peridic response of the (v_2, w_2)-components corresponding to the oscillation in the (v_1, w_1)-components.

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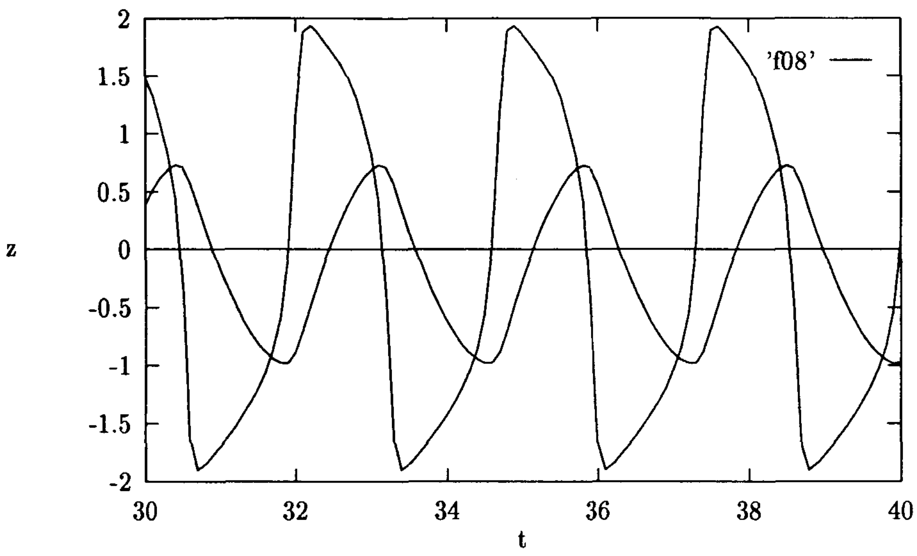

Figure 8: The (v_1, w_1)-components of a numerical solution. The graphs $z = v_1(t)$ and $z = w_1(t)$ are shown in the (t, z)-plane. Here, $v_1 = v_1(t)$ and $w_1 = w_1(t)$ are the components of the solution whose trajectory in the (v_1, w_1)-plane is shown in Figure 7. $v_1(30)$ and $w_1(30)$ are given approximately by $v_1(30) \approx 1.495249$ and $w_1(30) \approx 0.393799$.

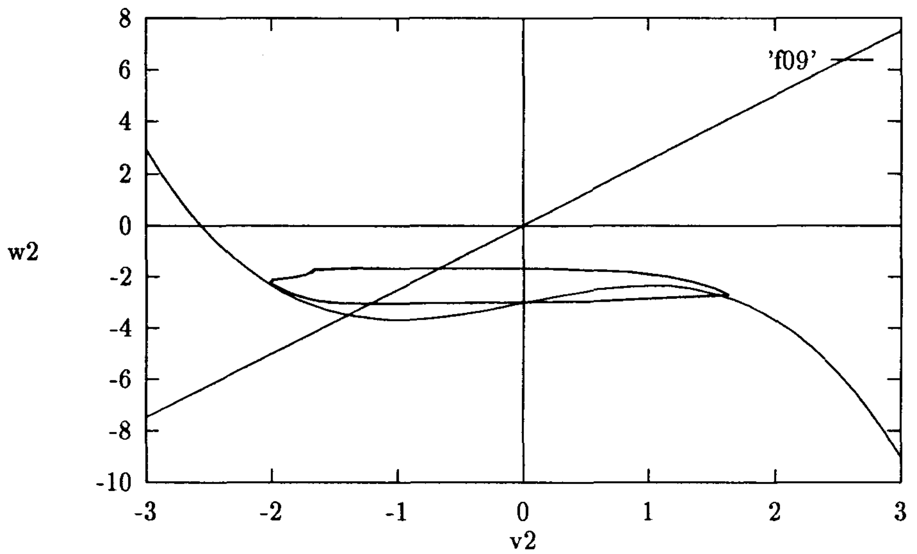

Figure 9: The trajectory of a solution of the coupled system. The (v_2, w_2)-components of the trajectory of a numerical solution of the system at (14) with the values of the parameters given at (12), (13), and (15) is shown in the (v_2, w_2)-plane. The nulclines $w_2 = v_2 - v_2^3/3 + a_2$ and $w_2 = (v_2 + c_2)/b_2$ are also shown.

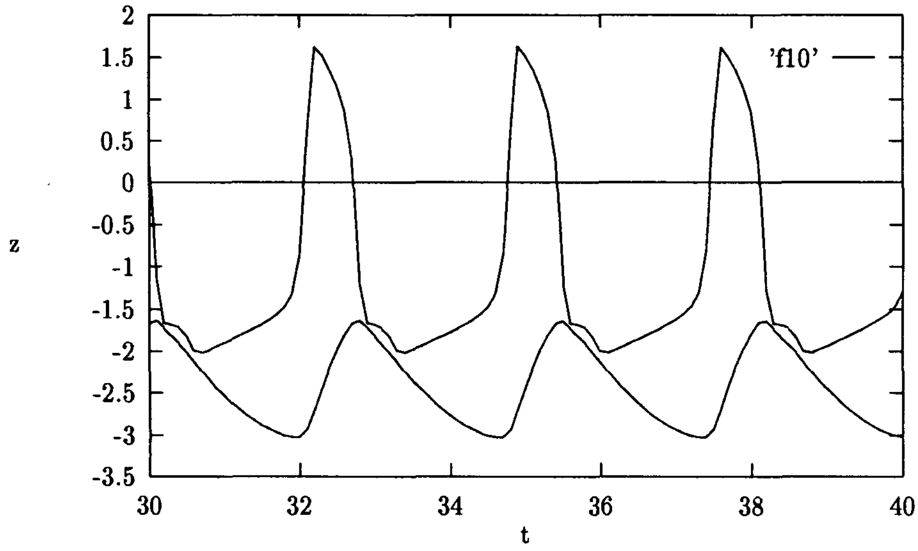

Figure 10: The (v_2, w_2)-components of a numerical solution. The graphs $z = v_2(t)$ and $z = w_2(t)$ are shown in the (t, z)-plane. Here, $v_2 = v_2(t)$ and $w_2 = w_2(t)$ are the components of the solution whose trajectory in the (v_2, w_2)-plane is shown in Figure 9. $v_1(30)$ and $w_1(30)$ are given approximately by $v_2(30) \approx 0.334839$ and $w_2(30) \approx -1.668253$.