The Resolution Modules of A Space and Its Universal Covering Space

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Let $G$ be a finite group, $Y$ a finite connected $G$-CW-complex, and let $\varPi(Y)$ denote the $G$-poset (in the sense of Oliver-Petrie) associated to $Y$. They defined the abelian group $\varTheta(G, \varPi(Y))$ consisting of all equivalent classes of $\varPi(Y)$-complexes. They also defined the subgroup $\varPhi(G, \varPi(Y))$ related to $\varPi(Y)$-resolutions. We call $\varPhi(G, \varPi(Y))$ the resolution module of $Y$. Applying the Oliver-Petrie theory to the universal covering space $\check{Y}$, we obtain the group $\varTheta(G, \varPi(\check{Y}))$, where $G$ is a certain extension of $G$ by $\varPi_1(Y)$. Then the canonical homomorphism $\nu : \varTheta(G, \varPi(\check{Y})) \to \varTheta(G, \varPi(Y))$ induced by the projection $\check{Y} \to Y$ is an isomorphism. In this paper, for $G = \mathbb{Z}_p \times \mathbb{Z}_q$ we construct a finite $G$-CW-complex $Y$ such that $\varPi_1(Y) \cong \mathbb{Z}_p$ and $\nu(\varPhi(G, \varPi(Y))) \neq \varPhi(G, \varPi(Y))$, where $p$ and $q$ are arbitrary distinct primes.

Keywords: $G$-CW-complex, $G$-map, $G$-poset

1 INTRODUCTION

Throughout this paper let $G$ be a finite group and $S(G)$ denote the set of all subgroups of $G$. Let $f : X \to Y$ be a $G$-map between finite $G$-CW-complexes. When does there exist a $G$-CW-complex $X' \supseteq X$ with $X'^G = X^G$ and a quasi-equivalence $f' : X' \to Y$ extending $f$? Here a quasi-equivalence $f' : X' \to Y$ means that $f'$ is a $G$-map inducing an isomorphism on $\varPi_1$ and integral homology. R.Oliver and T.Petrie treated this problem in [5]. To solve the problem, they introduced the set

$$\varPi(Y) = \coprod_{H \in S(G)} \varPi_0(Y^H)$$

(the disjoint union of $\varPi_0(Y^H)$'s).

Here $Y^H$ is the $H$-fixed point set of $Y$ and $\varPi_0(Y^H)$ is the set of all connected components of $Y^H$. The set $\varPi(Y)$ is called a $G$-poset associated to $Y$. We regard $S(G)$ as a $G$-set via the action $(g, H) \mapsto gHg^{-1}$ ($g \in G$ and $H \in S(G)$) and as a partially ordered set via

$$H < K \iff H \supseteq K \quad (H, K \in S(G)).$$

Let $S(Y)$ denote the set of all subcomplexes of $Y$. We also regard $S(Y)$ as a $G$-set by left translation, i.e. $(g, A) \mapsto gA$ ($g \in G$ and $A \in S(Y)$). Suppose that $S(G) \times S(Y)$ has the diagonal action, i.e. $(g, (H, A)) \mapsto (gHg^{-1}, gA)$ ($g \in G, H \in S(G), A \in S(Y)$).

For $\alpha \in \varPi(Y)$, there exists uniquely a subgroup $H \in S(G)$ such that $\alpha \in \varPi_0(Y^H)$. Hence we can define a map $\rho : \varPi(Y) \to S(G)$ by $\alpha \mapsto H$. In addition, $\varPi(Y)$ is given the partial order $\leq$ by

$$\alpha \leq \beta \iff \rho(\alpha) \supseteq \rho(\beta) \quad (\alpha, \beta \in \varPi(Y))$$

where $|\alpha|$ is the underlying space for $\alpha \in \varPi(Y)$.

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Definition 1.1. We abbreviate $\Pi(Y)$ to $\Pi$. A finite $G$-CW-complex $Z$ with a basepoint $q$ is called a $\Pi$-complex if it is equipped with a specified set $\{Z_\alpha \mid \alpha \in \Pi\}$ of subcomplexes $Z_\alpha$ of $Z$, satisfying the following four conditions:

(i) $q \in Z_\alpha$,
(ii) $gZ_\alpha = Z_{ga}$ for $g \in G$, $\alpha \in \Pi$,
(iii) $Z_\alpha \subseteq Z_\beta$ if $\alpha \leq \beta$ in $\Pi$, and
(iv) for any $H \in S(G)$, $Z^H = \bigvee_{\alpha \in \Pi \text{ with } \rho(\alpha) = H} Z_\alpha$.

Let $\mathcal{F}$ denote the family of all $\Pi$-complexes and define the equivalence relation $\sim$ on $\mathcal{F}$ by

$$Z \sim W \iff \chi(Z_\alpha) = \chi(W_\alpha) \quad \text{for all } \alpha \in \Pi \quad (Z, W \in \mathcal{F})$$

where $\chi(Z_\alpha)$ is the Euler characteristic of $Z_\alpha$.

The set $\Omega(G, \Pi) = \mathcal{F}/\sim$ is an abelian group via

$$[Z] + [W] = [Z \vee W] \quad (Z, W \in \mathcal{F}).$$

Moreover it is finitely generated. We call $\Omega(G, \Pi)$ the Oliver-Petrie module associated with $\Pi$.

The set $\Delta(G, \Pi) = \{[Z] \in \Omega(G, \Pi) \mid Z \text{ is contractible}\}$ is a submodule of $\Omega(G, \Pi)$. By [5, Proposition 2.6] the submodule $\Phi(G, \Pi)$ given below is useful for computing $\Delta(G, \Pi)$, since

$$\Phi(G, \Pi) \supset \Delta(G, \Pi) \quad \text{and} \quad [\Phi(G, \Pi) : \Delta(G, \Pi)] < \infty.$$ 

We define

$$\mathcal{P}(\Pi) = \{\alpha \in \Pi \mid \rho(\alpha) \text{ is a subgroup of } G \text{ of prime power order}\}, \quad \text{and}

S(G, \alpha) = \{K \in S(G) \mid \rho(\alpha) \cdot K \subseteq G_\alpha \text{ and } K/\rho(\alpha) \text{ is cyclic}\}$$

where $G_\alpha$ is the isotropy subgroup at $\alpha$. We set $\bar{\chi}(Z) = \chi(Z) - 1$ for any space $Z$. Then the resolution module $\Phi(G, \Pi)$ is defined by

$$\Phi(G, \Pi) = \{[Z] \in \Omega(G, \Pi) \mid \bar{\chi}((Z_\alpha)^K) = 0, \quad \text{for all } \alpha \in \mathcal{P}(\Pi) \text{ and } K \in S(G, \alpha)\}.$$ 

It is easy to check that $\Phi(G, \Pi)$ is a subgroup of $\Omega(G, \Pi)$. This $\Phi(G, \Pi)$ can be defined in the term of $\Pi$-resolutions, which will be explained in 2.3. Applying the Oliver-Petrie theory to a covering space, M.Morimoto and K.Iizuka [4] gave a necessary and sufficient condition to extend a $G$-map $f : X \to Y$ to a pseudo-equivalence $f'' : X'' \to Y$ such that $X''^G = X^G$ when $\pi_1(Y)$ is finite. Here a pseudo-equivalence $f''$ means a $G$-map which is a (non-equivariant) homotopy equivalence.

Let $G$ and $\tilde{G}$ be finite groups, $\sigma : \tilde{G} \to G$ an epimorphism, $Y$ a finite connected $G$-CW-complex, $\tilde{Y}$ a finite connected $\tilde{G}$-CW-complex, and $(\tilde{Y}, p, Y)$ a $\sigma$-equivariant covering space (i.e. $p(gb) = \sigma(g)p(b)$ for $g \in \tilde{G}$, $b \in \tilde{Y}$). Put $\pi = \ker \sigma$. Furthermore assume that $\pi$ acts freely and transitively on each fiber. Under the conditions, the canonical map $\nu : \Omega(\tilde{G}, \Pi(\tilde{Y})) \to \Omega(G, \Pi(Y))$ is defined by $[\tilde{X}] \mapsto [G \times_\sigma \tilde{X}]$ and it is an isomorphism. As for the resolution submodules, we have $\nu(\Delta(\tilde{G}, \Pi(\tilde{Y}))) \subseteq \Delta(G, \Pi(Y))$ and $\nu(\Phi(\tilde{G}, \Pi(\tilde{Y}))) \subseteq \Phi(G, \Pi(Y))$ [4, Proposition 3.6]. In the present paper, we study the next problem:

**Problem** Do there exist $G$-CW-complexes $Y$ such that

$$\nu(\Phi(\tilde{G}, \Pi(\tilde{Y}))) \neq \Phi(G, \Pi(Y))?$$

Our result is:

**Theorem 1.2.** Let $p$, $q$ be distinct primes, $G = \mathbb{Z}_p \times \mathbb{Z}_q$ and $\tilde{G} = \pi \times (\mathbb{Z}_p \times \mathbb{Z}_q)$, where $\pi$ is a copy of $\mathbb{Z}_p$. Then there exists a finite connected and simply connected $\tilde{G}$-CW-complex $\tilde{Y}$ such that the $G$-CW-complex $Y = \tilde{Y}/\pi$ satisfies $\pi_1(Y) \cong \pi$ and $\nu(\Phi(\tilde{G}, \Pi(\tilde{Y}))) \neq \Phi(G, \Pi(Y))$. 

This paper is organized as follows. In Section 2, we review basic properties of the Oliver-Petrie module and the resolution module. In Section 3, we study relations between the posets of a base space and its covering space. Finally, in Section 4, we prove Theorem 1.2.
2 BASIC PROPERTIES OF THE OLIVER-PETRIE MODULES

In this section, we recall basic properties needed later from R.Oliver-T.Petrie [5] and M.Morimoto-K.Iizuka [4].

2.1 For a finite \( G \)-CW-complex \( Y \), the map \( \rho \times | : \Pi(Y) \to S(G) \times S(Y) \) given by \( \alpha \mapsto (\rho(\alpha), |(\alpha)) \) is injective. We regard \( \Pi(Y) \) as a subset of \( S(G) \times S(Y) \). Then \( \Pi = \Pi(Y) \) has a \( G \)-action given by \( (g, \alpha) \mapsto g(\rho \times |)(\alpha) \). Furthermore \( \Pi \) satisfies the following three conditions:

(i) \( \rho(\alpha) \leq G_\alpha \) for \( \alpha \in \Pi \),
(ii) if \( \alpha \leq \beta \) then \( g\alpha \leq g\beta \) for \( g \in G \), and
(iii) for \( \alpha \in \Pi \) and \( H \leq \rho(\alpha) \), there exists uniquely \( \gamma \in \Pi \) such that \( \gamma \geq \alpha \) and \( \rho(\gamma) = H \).

In the case where \( Y = \{\ast\} \) (a singleton),

\[
\Pi(Y) = \prod_{H \in S(G)} \pi_0(\{\ast\}^H)^{\rho(\ast)} \prod_{H \in S(G)} \{(H, \{\ast\})\}^{proj} S(G).
\]

Let \( Z \) be a \( \Pi \)-complex. For each cell \( c \) in \( Z \setminus \{\ast\} \), there exists a unique element \( \alpha(c) \in \Pi \) such that \( \rho(\alpha(c)) = G_x \), \( x \in c \), and \( c \in Z(c) \). We say that \( c \) of type \( \alpha(c) \).

2.2 For each \( \alpha \in \Pi(Y) \), the \( G \)-space \( \{\ast\}^\alpha = G/\rho(\alpha) \Pi \{\ast\} \) is equipped with \( \Pi(Y) \)-complex structure such that

\[
\{\ast\}^\beta = \{gp(\alpha) \mid g \in G, \ g\alpha \leq \beta\} \Pi \{\ast\} \quad \text{for} \ \beta \in \Pi(Y).
\]

Let \( \{\alpha_i \mid 1 \leq i \leq s\} \) be the complete representative system of \( \Pi(Y)/G \). Then the set \( \Omega(G, \Pi(Y)) \) is a free abelian group with a basis \( \{[\{\alpha_i\}]^+ \mid 1 \leq i \leq s\} \) i.e.

\[
\Omega(G, \Pi(Y)) = \langle [\{\alpha_i\}]^+ \mid 1 \leq i \leq s \rangle_Z.
\]

Suppose hereafter that \( Y \) is a finite connected \( G \)-CW-complex. Then \( \pi_0(Y^{(1)}) \) consists of a unique element which will be denoted by \( m \). The element \( m \) is the maximal element in \( \Pi(Y) \).

2.3 A finite \( k \)-dimensional \( \Pi(Y) \)-complex \( Z \) is called a \( \Pi(Y) \)-resolution if \( Z \) satisfies the following three conditions:

(i) \( Z \) is connected and simply-connected,
(ii) \( Z \) is \( (k-1) \)-connected, and
(iii) \( H_k(Z; Z) \) is \( Z[G] \)-projective.

If \( Z \) is a \( k \)-dimensional \( \Pi(Y) \)-resolution, set

\[
\gamma_G(Z) = (-1)^k[H_k(Z; Z)] \in K_0(Z[G]),
\]

where \( K_0(Z[G]) \) is the Grothendieck group of finitely generated projective \( Z[G] \)-modules modulo free modules.

For a \( \Pi(Y) \)-resolution \( Z \), we get a \( \Pi(Y) \)-complex \( Z^* \) with \( \chi(Z^*) = 0 \) by attaching some free cells \( G \times D^i \) to \( Z \). Clearly \( \chi(Z^*) = \chi(Z) \) for any \( \alpha \in \Pi(Y) \setminus \{m\} \). Moreover for a \( k \)-dimensional \( \Pi(Y) \)-resolution \( Z \) with \( k \geq 1 \), there exists a \( \Pi(Y) \)-resolution \( W \) satisfying the following conditions:

(i) \( \dim W = k + 1 \),
(ii) \( \gamma_G(Z) = \gamma_G(W) \), and
(iii) \( [Z^*] = [W^*] \) in \( \Omega(G, \Pi(Y)) \).

By [5, Proposition 2.6], \( \Phi(G, \Pi(Y)) \) defined in Section 1 coincides with \( \{[Z^*] \in \Omega(G, \Pi(Y)) \mid Z \text{ is a } \Pi(Y) \text{-resolution}\} \).

Example 2.4. Let \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( Y = \{\ast\} \) (a singleton). There are three subgroups isomorphic to \( \mathbb{Z}_2 \). We denote them by \( \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3 \). By 2.1,

\[
\Pi(\{\ast\}) = S(G) = \{\{1\}, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}.
\]

The partially ordered set \( \Pi(\{\ast\}) \) is illustrated by the diagram below. We arrange the elements of \( \Pi(\{\ast\}) \) such that if \( a > b \) \((a, b \in \Pi(\{\ast\}))\), then \( a \) is situated above \( b \). Furthermore we connect \( a \) and \( b \) by a
Since $G$ is of prime power order, $\mathcal{P}(\Pi(\{\ast\}))$ coincides with $\Pi(\{\ast\})$. As $G$ is abelian, the $G$-action on $\Pi(\{\ast\}) = S(G)$ is trivial, which amounts to
\[ \Pi(\{\ast\})/G = S(G)/G = S(G). \]

By 2.2, the free abelian group $\Omega(G, \Pi(Y))$ has the basis
\[ \{ ([1])^+, ([Z_2])^+, ([Z_2^2])^+, ([Z_2^3])^+, ([Z_2 \times Z_2])^+ \}. \]

In the following, we show that $\Phi(G, \Pi(\{\ast\}))$ is the trivial group. Each $[Z] \in \Phi(G, \Pi(\{\ast\}))$ is uniquely written in the form:
\[ [Z] = n_{Z_2} x_{Z_2} ([Z_2 \times Z_2])^+ + n_{Z_2^2} ([Z_2^2])^+ + n_{Z_2^3} ([Z_2^3])^+ + n_{Z_2^1} ([Z_2^1])^+ + n_{\{1\}} ([\{1\}])^+, \]
where each coefficient is some integer and satisfies the condition
\[
\tilde{x}(Z^K) = n_{Z_2} x_{Z_2} \tilde{x}(([Z_2 \times Z_2])^+) + n_{Z_2^2} \tilde{x}(([Z_2^2])^+K) + n_{Z_2^3} \tilde{x}(([Z_2^3])^+K) + n_{Z_2^1} \tilde{x}(([Z_2^1])^+K) + n_{\{1\}} \tilde{x}(([\{1\}])^+K) = 0 \tag{2.4.1}
\]
for each $\alpha \in \mathcal{P}(\Pi(\{\ast\}))$ and $K \in S(G, \alpha)$. Using (2.4.1), we shall verify that all coefficients vanish.

First, consider the case of $\alpha = Z_2^1$. Then we have $S(G, \alpha) = \{Z_2^1, Z_2^2 \times Z_2\}$. For $\alpha = Z_2^1$ and $K = Z_2^1$, since
\[
\tilde{x}((Z_2 \times Z_2)_{Z_2^1}^{Z_2^1}) = \tilde{x}((Z_2^1) \{\ast\}) = 1, \\
\tilde{x}((Z_2^2)_{Z_2^1}^{Z_2^1}) = \tilde{x}((Z_2^2) \{\ast\}) = 2, \quad \text{and} \\
\tilde{x}((Z_2^3)_{Z_2^1}^{Z_2^1}) = \tilde{x}((Z_2^3) \{\ast\}) = \tilde{x}((Z_2^1) \{\ast\}) = 0,
\]
the equation (2.4.1) implies
\[ n_{Z_2} x_{Z_2} + 2n_{Z_2^1} = 0. \tag{2.4.2} \]

Next for $\alpha = Z_2^2$ and $K = Z_2^1 \times Z_2$, since
\[
\tilde{x}((Z_2 \times Z_2)_{Z_2^1}^{Z_2^1}Z_2) = \tilde{x}((Z_2 \times Z_2) \{\ast\}) = 1, \quad \text{and} \\
\tilde{x}((Z_2^1)_{Z_2^1}^{Z_2^1}Z_2) = \tilde{x}((Z_2^1) \{\ast\}) = \tilde{x}((Z_2^1)_{Z_2^1}^{Z_2^1}Z_2) = \tilde{x}((Z_2^1) \{\ast\}) = 0,
\]
the equation (2.4.1) implies
\[ 2n_{Z_2^1} x_{Z_2} = 0. \]
we obtain
\[ n_{\mathbb{Z}_2 \times \mathbb{Z}_2} = 0. \tag{2.4.3} \]

We get \( n_{\mathbb{Z}_1} = 0, \ n_{\mathbb{Z}_2 \times \mathbb{Z}_2} = 0 \) by (2.4.2) and (2.4.3). Similarly for \( \alpha = \mathbb{Z}_2^{2} \) and \( \mathbb{Z}_3^{2} \), we have \( n_{\mathbb{Z}_2^{2}} = 0 \) and \( n_{\mathbb{Z}_3^{2}} = 0 \). Moreover the case where \( \alpha = \{1\} \), we have
\[ S(G, \alpha) = \{\{1\}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3\}. \]

Particularly, in the case where \( \alpha = \{1\}, \ K = \{1\} \), we have
\[ 0 = n_{\{1\} \times \mathbb{Z}} = n_{\{1\}} = \chi(G) = 4n_{\{1\}}. \]

Hence \( n_{\{1\}} = 0 \). Putting all together,
\[ n_{\mathbb{Z}_2 \times \mathbb{Z}_2} = n_{\mathbb{Z}_1} = n_{\mathbb{Z}_2^{2}} = n_{\mathbb{Z}_3^{2}} = n_{\{1\}} = 0. \]

This concludes \([Z] = 0\).

3 RELATIONS BETWEEN THE POSETS OF A BASE SPACE AND ITS COVERING SPACE

In this section let \( G \) and \( \widetilde{G} \) be finite groups, \( \sigma : \widetilde{G} \to G \) an epimorphism, \( Y \) a finite connected \( G \)-CW-complex, \( \widetilde{Y} \) a finite connected \( \widetilde{G} \)-CW-complex, and \( p : \widetilde{Y} \to Y \) a \( \sigma \)-equivariant covering space. We put \( \pi = \ker \sigma \). Moreover we assume that \( \pi \) acts freely and transitively on each fiber. Remark that the \( G \)-action on \( Y \) gives a \( G \)-poset \( \Pi(Y) \) and a \( G \)-map \( p : \Pi(Y) \to S(G) \).

Let \( \alpha \) be an element of \( \Pi(Y) \). Then \( |\alpha| \) is a connected component of \( \widetilde{Y}p(\alpha) \). Hence \( p(|\alpha|) \) is connected. Moreover we have \( p(|\alpha|) \subseteq Y^{p(\alpha)} \). Thus there exists a unique connected component \( \alpha \in \Pi(Y) \) such that \( p(\alpha) = \sigma(p(\alpha)) \) and \( |\alpha| \supseteq p(|\alpha|) \). Now we define the map \( \mu : \Pi(\widetilde{Y}) \to \Pi(Y) \) by \( \widetilde{\alpha} \mapsto \alpha \).

**Lemma 3.1.** In the above situation, \( \rho(\mu(\alpha)) = \sigma(p(\alpha)) \) and \( |\mu(\alpha)| = p(|\alpha|) \) hold for any \( \alpha \in \Pi(\widetilde{Y}) \).

**Proof.** We have already showed \( \rho(\mu(\alpha)) = \sigma(p(\alpha)) \). It suffices to show that \( |\alpha| \supseteq p(|\alpha|) \), where \( \alpha = \mu(\widetilde{\alpha}) \). First we take \( \widetilde{y}_0 \in |\widetilde{\alpha}| \), and set \( y_0 = p(\widetilde{y}_0) \). Take \( y_1 \in |\alpha| \) arbitrarily. Remark that \( y_0 \in |\widetilde{\alpha}| \) and \( y_1 \in |\alpha| \).

Then there exists a path \( y(t) : I \to |\alpha| \) such that \( y(0) = y_0 \) and \( y(1) = y_1 \), where \( I = [0,1] \). Then we have a lift \( \widetilde{y}(t) : I \to \widetilde{Y} \) of \( y(t) \) with \( \widetilde{y}(0) = \widetilde{y}_0 \). On the other hand, for any \( \widetilde{y} \in \widetilde{p}(\alpha) \), a path \( \widetilde{g} \widetilde{y}(t) : I \to \widetilde{Y} \) is also a lift of \( y(t) \) with \( \widetilde{g} \widetilde{y}(0) = \widetilde{y}_0 \). Hence we have \( \widetilde{g} \widetilde{y}(t) = \widetilde{g}(t) \) for any \( \widetilde{g} \in \widetilde{p}(\alpha) \). It follows at once that \( \widetilde{y}(1) \in \widetilde{p}(\alpha) \). Since \( \widetilde{y}_0 \in |\alpha| \subseteq \widetilde{p}(\alpha) \), we have \( \widetilde{y}(1) \in |\alpha| \). Thus \( y_1 = p(\widetilde{y}(1)) \in p(|\alpha|) \). This means that \( |\alpha| \supseteq p(|\alpha|) \).

By Lemma 3.1, the following diagram commutes:
\[ \begin{array}{ccc}
\Pi = \Pi(Y) & \xrightarrow{\rho \times |} & S(G) \times S(Y) \\
\mu \downarrow & & \downarrow \sigma \times p \\
\Pi = \Pi(Y) & \xrightarrow{\rho \times |} & S(G) \times S(Y).
\end{array} \]

**Proposition 3.2.** For any \( \alpha \in \Pi(Y), \mu^{-1}(\alpha) \) is non-empty. Moreover \( \pi \) acts transitively on \( \mu^{-1}(\alpha) \).

**Proof.** We first show that for any \( \alpha \in \Pi(Y), \mu^{-1}(\alpha) \) is non-empty. Arbitrarily choose and fix \( y \in |\alpha| \). Since \( p : \widetilde{Y} \to Y \) is surjective, there exists \( \widetilde{y} \in p^{-1}(y) \). Now, remark that \( \sigma| \widetilde{G}_{\widetilde{y}} : \widetilde{G}_{\widetilde{y}} \to G_{y} \) is an isomorphism. Since \( y \in |\alpha| \subseteq Y^{p(\alpha)} \), we have \( \rho(\alpha) \subseteq G_{y} \). Put \( \widetilde{H} = (\sigma| \widetilde{G}_{\widetilde{y}})^{-1}(\rho(\alpha)) \). Since \( \widetilde{H} \subseteq \widetilde{G}_{\widetilde{y}} \), \( \widetilde{y} \) lies in \( \widetilde{Y}^{\widetilde{H}} \). Hence there exists \( \alpha \in \pi_{0}(\widetilde{Y}^{\widetilde{H}}) \) with \( \widetilde{y} \in |\alpha| \), which implies \( \rho(\alpha) = \widetilde{H} \). Thus we obtain \( \rho(\mu(\alpha)) = \sigma(p(\alpha)) = \sigma(\widetilde{H}) = \rho(\alpha), \ y = p(\widetilde{y}) \in p(|\alpha|) = |\mu(\alpha)|, \) and \( y \in |\mu(\alpha)| \cap |\alpha| \neq \emptyset \). It follows at once that \( \mu(\alpha) = \alpha \). Namely, \( \mu^{-1}(\alpha) \) is non-empty.
Next we shall prove that $\pi (= \ker \sigma)$ acts transitively on $\mu^{-1}(a)$. Let $\tilde{a}$ and $\tilde{b}$ be elements of $\mu^{-1}(a)$. It suffices to show that $\tilde{a} = \tilde{b}$ for some $\tilde{h} \in \pi$. By the definition of $\mu$, we have $\sigma(\tilde{p}(\tilde{a})) = \rho(\tilde{a}) = \sigma(\tilde{p}(\tilde{b}))$ and $p(|\tilde{a}|) = |\tilde{h}| = p(|\tilde{b}|)$. Let $\tilde{a}$ and $\tilde{b}$ be the points on $|\tilde{a}|$ and $|\tilde{b}|$ respectively such that $p(\tilde{a}) = y = p(\tilde{b})$. Then there exists $\tilde{h} \in \pi$ such that $\tilde{h} \tilde{a} = \tilde{b}$ because $\pi$ acts transitively on each fiber. Now, it should be noted that $\tilde{p}(\tilde{a}) \subseteq \tilde{G}_{\tilde{a}}$ and $\tilde{p}(\tilde{b}) \subseteq \tilde{G}_{\tilde{b}}$. Observe that $\tilde{G}_{\tilde{a}} \equiv \tilde{G}_{\tilde{b}} = \tilde{h} \tilde{G}_{\tilde{a}} \tilde{h}^{-1}$. Moreover since $\tilde{p}(\tilde{b}) \subseteq \tilde{G}_{\tilde{b}}$, we have $\sigma(\tilde{p}(\tilde{b})\tilde{h}^{-1}) = \sigma(\tilde{h})\sigma(\tilde{p}(\tilde{a}))\sigma(\tilde{h}^{-1}) = p(\tilde{a})$. Recalling that $\sigma(\tilde{p}(\tilde{b})) = p(\tilde{a})$, we get $\tilde{h} \tilde{p}(\tilde{a}) \tilde{h}^{-1} = \tilde{p}(\tilde{b})$, that is, $\tilde{p}(\tilde{h} \tilde{a}) = \tilde{p}(\tilde{b})$. Therefore we have $\tilde{h} \tilde{a} = \tilde{b} \in \tilde{p}(\tilde{Y}(\tilde{p}(\tilde{b})))$. Remark that $\tilde{b} = \tilde{h} \tilde{a} \in |\tilde{a}| = |\tilde{h} \tilde{a}|$. It follows at once that $\tilde{b} \in |\tilde{h} \tilde{a}| \cap |\tilde{b}| \neq \emptyset$. Thus $\tilde{h} \tilde{a} = \tilde{b}$.

Henceforth let $\{\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_s\}$ be a complete representative system of $\Pi(\tilde{Y})/$\tilde{G}, that is,

$$\Pi(\tilde{Y}) = \prod_{i=1}^{s} \tilde{G} \tilde{a}_i \text{ (disjoint union)}.$$

**Lemma 3.3.** For $i \neq j$, one has $\mu(\tilde{G} \tilde{a}_i) \cap \mu(\tilde{G} \tilde{a}_j) = \emptyset$.

**Proof.** Suppose that $\mu(\tilde{G} \tilde{a}_i) \cap \mu(\tilde{G} \tilde{a}_j) \ni \alpha$. Then $\alpha$ is written in two ways: $\alpha = \mu(\tilde{g}_1 \tilde{a}_i) = \mu(\tilde{g}_2 \tilde{a}_j)$ for $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$. Since $\mu^{-1}(\alpha) \ni \tilde{g}_1 \tilde{a}_i, \tilde{g}_2 \tilde{a}_j$, by Proposition 3.2 there exists $\tilde{h} \in \pi$ such that $\tilde{g}_1 \tilde{a}_i = \tilde{h} \tilde{g}_2 \tilde{a}_j$. This means $\tilde{g}_1 \tilde{a}_i \in \tilde{G} \tilde{a}_i \cap \tilde{G} \tilde{a}_j$, so we get a contradiction. \hfill \square

Next we shall show that $\mu$ is a $\sigma$-equivariant map.

**Lemma 3.4.** For $\tilde{g} \in \tilde{G}$, $\tilde{a} \in \Pi(\tilde{Y})$, one has $\mu(\tilde{g} \tilde{a}) = \sigma(\tilde{g})\mu(\tilde{a})$.

**Proof.** It suffices to show that $(\rho \times 1)(\mu(\tilde{g} \tilde{a})) = (\rho \times 1)(\sigma(\tilde{g})\mu(\tilde{a}))$. The following hold:

$$\begin{align*}
\rho(\mu(\tilde{g} \tilde{a})) &= \sigma(\tilde{p}(\tilde{g} \tilde{a})) \\
&= \sigma(\tilde{g} \sigma(\tilde{p}(\tilde{a}))\tilde{g}^{-1}) \\
&= \sigma(\tilde{g})\sigma(\tilde{p}(\tilde{a}))\sigma(\tilde{g})^{-1} \\
&= \sigma(\tilde{g})\rho(\mu(\tilde{a}))\sigma(\tilde{g})^{-1} \\
&= \rho(\sigma(\tilde{g})\mu(\tilde{a})), \\
|\mu(\tilde{g} \tilde{a})| &= p(|\tilde{g} \tilde{a}|) \\
&= p(\tilde{g} |\tilde{a}|) \\
&= |\sigma(\tilde{g})| |\mu(\tilde{a})| \\
&= |\sigma(\tilde{g})\mu(\tilde{a})|.
\end{align*}$$

Hence we have

$$(\rho \times 1)(\mu(\tilde{g} \tilde{a})) = (\rho \times 1)(\sigma(\tilde{g})\mu(\tilde{a})). \hfill \square$$

Using Lemmas 3.3 and 3.4, we show that $\Omega(\tilde{G}, \Pi(\tilde{Y}))$ and $\Omega(G, \Pi(Y))$ are abstractly isomorphic.

**Proposition 3.5.** Both $\Omega(\tilde{G}, \Pi(\tilde{Y}))$ and $\Omega(G, \Pi(Y))$ have the same rank.

**Proof.** Note that $\mu$ is surjective by Proposition 3.2. We have the following:

$$\begin{align*}
\Pi(Y) &= \mu(\Pi(\tilde{Y})) \\
&= \mu(\prod_{i=1}^{s} \tilde{G} \tilde{a}_i) \\
&= \prod_{i=1}^{s} \mu(\tilde{G} \tilde{a}_i) \\
&= \prod_{i=1}^{s} \sigma(\tilde{G})\mu(\tilde{a}_i) \\
&= \prod_{i=1}^{s} G \mu(\tilde{a}_i).
\end{align*}$$
Thus \( \{ \mu(\tilde{a}_1), \mu(\tilde{a}_2), \ldots, \mu(\tilde{a}_s) \} \) is a complete representative system of \( \Pi(Y)/G \). By 2.2, rank \( (\Omega(\tilde{G}, \Pi(Y))) \) coincides with rank \( (\Omega(G, \Pi(Y))) \).

In the remainder of this section, we shall show that the canonical map \( \nu: \Omega(\tilde{G}, \Pi(Y)) \to \Omega(G, \Pi(Y)) \) is an isomorphism.

**Definition 3.6.** Given a \( \tilde{G} \)-space \( \tilde{X} \), let \( (g, \tilde{x}), (g', \tilde{z}) \in G \times \tilde{X} \). Then we write \( (g, \tilde{x}) \sim (g', \tilde{z}) \) to mean that there exists \( \tilde{g} \in \tilde{G} \) such that \( g' = g \sigma(\tilde{g})^{-1}, \tilde{z} = \tilde{g} \tilde{x} \). This relation \( \sim \) can be easily verified to be an equivalence relation. The quotient space \( (G \times \tilde{X})/\sim \) is denoted by \( \tilde{X} \times_\sigma \tilde{X} \).

Remark that \( G \)-action on \( \tilde{X} \times_\sigma \tilde{X} \) is naturally defined by \( (g', [g, \tilde{x}]) \mapsto [g'g, \tilde{x}] \) for \( g', g \in G \), and \( \tilde{x} \in \tilde{X} \). We regard \( \tilde{X} \times_\sigma \tilde{X} \) as a \( G \)-space with respect to this action.

Suppose that \( X \) has a \( \Pi(Y) \)-complex structure \( (X, \{ X_a \}_{a \in \Pi(Y)}) \). Setting \( X = \tilde{G} \times_\sigma \tilde{X} \), we define the map \( \tilde{X}_\alpha \): \( \tilde{X} \to X \) by \( \tilde{X}_\alpha(\tilde{x}) = [1, \tilde{x}] \). Take the point of \( X \) to which \( \tilde{X}_\alpha \) maps the basepoint of \( \tilde{X} \). For \( \alpha \in \Pi(Y) \), we define \( X_{\alpha\tilde{X}} = \bigcup_{\tilde{a} \in \mu^{-1}(\alpha)} p_{\tilde{X}}(\tilde{X}_{\tilde{a}}) \).

Let \( \tilde{X}_\alpha \) be an element of \( \mu^{-1}(\alpha) \). Then \( X_{\alpha\tilde{X}} = p_{\tilde{X}}(\tilde{X}_{\tilde{a}}) \) holds. Indeed, it is easy to see that \( p_{\tilde{X}} \) is \( \sigma \)-equivariant. For \( \gamma \in \Pi(Y) \), we define \( \tilde{X}_\gamma = \bigcup_{\tilde{a} \in \Pi(\gamma)} p_{\tilde{X}}(\tilde{X}_{\tilde{a}}) \).

We need the next lemma to prove Lemma 3.8, and Proposition 3.10 will follow from Lemmas 3.8 and 3.9.

**Lemma 3.7.** For \( \alpha, \gamma \in \Pi(Y) \) such that \( |\alpha| \cap |\beta| = 0 \), one has \( \tilde{X}_\alpha \cap \tilde{X}_\beta = \{ \} \).

**Proof.** Suppose that \( \tilde{X}_\alpha \cap \tilde{X}_\beta \neq \{ \} \). Then we can take a cell \( \tilde{e} \subseteq (\tilde{X}_\alpha \cap \tilde{X}_\beta) \setminus \{ \} \) and a point \( \tilde{x} \in \tilde{e} \). Let \( \tilde{g} \in \Pi(\gamma) \) be the type of \( \tilde{e} \). By 2.1, \( p(\tilde{g}) = \tilde{G} \) and \( \tilde{X}_\gamma \supset \tilde{e} \) hold. On the other hand, \( \tilde{x} \in \tilde{X}_\alpha \setminus \{ \} \subseteq \tilde{X}_\gamma \).

Hence we have \( p(\tilde{a}) \subseteq \tilde{G} \) and \( \tilde{Y}_\alpha(\tilde{a}) \subseteq \tilde{Y}_\gamma(\tilde{a}) \). For each \( \tilde{a}' \in \pi(\tilde{Y}_\alpha(\tilde{a})) \), there exists a unique \( \tilde{a}' \in \pi(\tilde{Y}_\gamma(\tilde{a})) \) such that \( \tilde{a}' \leq \tilde{a} \). Thus we obtain the map \( f: \pi(\tilde{Y}_\alpha(\tilde{a})) \to \pi(\tilde{Y}_\gamma(\tilde{a})) \) such that \( \tilde{a}' \leq f(\tilde{a}) \) for any \( \tilde{a}' \in \pi(\tilde{Y}_\alpha(\tilde{a})) \). If \( f(\tilde{a}) \neq \tilde{a} \), then by Definition 1.1(iv),

\[
\tilde{X}_{f(\tilde{a})(\gamma)} \cap \tilde{X}_\alpha = \{ \}.
\]

On the other hand, since \( \tilde{a} \leq \tilde{a}' \), we have \( \tilde{X}_\alpha \subseteq \tilde{X}_{\tilde{a}'(\gamma)} \), and hence

\[
\tilde{X}_{f(\tilde{a})(\gamma)} \cap \tilde{X}_\alpha \supseteq \tilde{X}_\alpha \cap \tilde{X}_\gamma \supseteq \tilde{e}.
\]

This is a contradiction, which concludes \( f(\tilde{a}) = \tilde{a} \). This implies \( \tilde{a} \leq \tilde{a} \). By an argument similar to the above, we have \( \tilde{a} \leq \tilde{a} \). Then since \( |\alpha| \subseteq |\alpha| \cap |\beta| \subseteq |\beta| \), \( |\alpha| \cap |\beta| \) contains \( |\gamma| \), which is not empty. This contradicts the assumption that \( |\alpha| \cap |\beta| = \emptyset \).

**Lemma 3.8.** For \( \alpha, \beta \in \pi_0(Y^H) \) such that \( \alpha \neq \beta \), one has \( X_\alpha \cap X_\beta = \{ \} \).

**Proof.** Let \( \tilde{a} \in \Pi(Y) \) be an element of \( \mu^{-1}(\gamma) \) for each \( \gamma \in \pi_0(Y^H) \). As noted previously, \( X_\alpha = p_{\tilde{X}}(\tilde{X}_{\tilde{a}}) \) and \( X_\beta = p_{\tilde{X}}(\tilde{X}_{\tilde{b}}) \). Suppose that \( X_\alpha \cap X_\beta \neq \{ \} \). We take \( x \in (X_\alpha \cap X_\beta) \setminus \{ \} \). Then \( x \) is written in two ways:

\[
x = p_{\tilde{X}}(\tilde{a}) = p_{\tilde{X}}(\tilde{b}),
\]

where \( \tilde{a} \in \tilde{X}_\alpha \setminus \{ \} \) and \( \tilde{b} \in \tilde{X}_\beta \setminus \{ \} \). Now, by the definition of \( p_{\tilde{X}} \), there exists \( \tilde{h} \in \pi \) with \( \tilde{h}\tilde{a} = \tilde{b} \). Since \( \tilde{a} \in \tilde{X}_\alpha \), we have \( \tilde{b} = \tilde{h}\tilde{a} \in \tilde{h}\tilde{X}_\alpha \setminus \{ \} \), hence \( \tilde{b} \in (\tilde{X}_{\tilde{h}\tilde{a}} \cap \tilde{X}_{\tilde{b}}) \setminus \{ \} \). Moreover by Lemma 3.7, since \( \tilde{h}\tilde{a} \cap \tilde{b} \neq \emptyset \), we have \( |\alpha| \cap |\beta| = p(\tilde{h}\tilde{a}) \cap p(\tilde{b}) \supseteq p(\tilde{h}\tilde{a} \cap \tilde{b}) \neq \emptyset \). Both \( \alpha \) and \( \beta \) are connected components of \( Y^H \), and so we obtain \( |\alpha| = |\beta| \), hence \( \alpha = \beta \). This is a contradiction, which implies \( X_\alpha \cap X_\beta = \{ \} \).

**Lemma 3.9.** For any subgroup \( H \) of \( G \),

\[
X^H = \bigcup_{\tilde{a} \in \Pi(\gamma)} p_{\tilde{X}}(\tilde{X}_{\tilde{a}}),
\]

s.t. \( p(\mu(\tilde{a})) = H \).
Proof. For each \( \tilde{a} \in \Pi(\tilde{Y}) \) with \( \rho(\mu(\tilde{a})) = H \), we have \( \sigma(\tilde{a}) = \rho(\mu(\tilde{a})) = H \) by definition. Since \( p\bar{X}(\tilde{X}^{\tilde{a}}) \subseteq X^{\rho(\tilde{a})} \) and \( \bar{X} \) is a \( \Pi(\tilde{Y}) \)-complex, we obtain \( p\bar{X}(\tilde{X}^{\tilde{a}}) \subseteq p\bar{X}(\tilde{X}^{\tilde{a}}) \subseteq X^{\rho(\tilde{a})} = H \).

Conversely, take \( x \in X^{H} \) arbitrarily. Since \( p\bar{X} \) is surjective, there exists \( x \in \pi_{X}^{-1}(x) \), and then we have \( \rho(x) = \tilde{a} \in p\bar{X}^{-1}(x) \), and then we have \( \rho(x) = \tilde{a} \in p\bar{X}^{-1}(x) \).

Proposition 3.10. The above space \( X \) is a \( \Pi(\tilde{Y}) \)-complex.

Proof. We must verify that \( X \) satisfies Definition 1.1(i)–(iv). Condition (i) is clearly fulfilled. We shall verify (ii)–(iv). First let \( \alpha \in \mu^{-1}(\tilde{a}) \) and \( \gamma \in \sigma^{-1}(\gamma) \). Then \( \rho(\gamma) = \sigma(\gamma) \rho(\tilde{a}) = \gamma \). This means \( \gamma \in \mu^{-1}(\alpha) \). Hence we have \( X_{\rho(\gamma)} = p\bar{X}(\tilde{X}^{\tilde{a}}) = p\bar{X}(\tilde{X}^{\tilde{a}}) = \sigma(\gamma) p\bar{X}(\tilde{X}^{\tilde{a}}) = \gamma X_{\rho(\gamma)} \), which verifies (ii).

Second, let \( \alpha \in \mu^{-1}(\tilde{a}) \) and \( \gamma \in \sigma^{-1}(\gamma) \). Take \( y \in [\tilde{a}] \) and set \( \gamma = p(y) \ (p([\tilde{a}]) = [\alpha] \subseteq Y^{\rho(\alpha)} \). By assumption, \( Y^{\rho(\alpha)} \subseteq Y^{\rho(\beta)} \). Hence we get \( \gamma \in Y^{\rho(\beta)} \). Then we have \( \rho(\beta) \subseteq G_{y} \). Recall \( \sigma|_{G_{y}} : G_{y} \to G_{y} \) is an isomorphism. Setting \( \tilde{H} = (\sigma|_{G_{y}})^{-1}(\rho(\beta)) \), we obtain an element \( \tilde{\beta} \in \Pi(\tilde{Y}) \) with \( [\tilde{\beta}] \supseteq [\tilde{a}] \). Since \( \rho(\tilde{\beta}) = \tilde{H} \subseteq \rho(\tilde{a}) \), we have \( \tilde{a} \subseteq \tilde{\beta} \). We get at once \( \sigma(\tilde{\beta}) = \tilde{H} \subseteq \rho(\tilde{a}) \), that is, \( \tilde{\beta} \in \mu^{-1}(\beta) \). Therefore \( X_{\alpha} = p\bar{X}(\tilde{X}^{\tilde{a}}) \subseteq p\bar{X}(\tilde{X}^{\tilde{a}}) \subseteq X_{\beta} \), finishes the verification of (iii). Finally Lemmas 3.8 and 3.9 guarantee (iv).

The next lemma will be used to prove Theorem 3.12.

Lemma 3.11. Let \( \tilde{a} \) be an element of \( \Pi(\tilde{Y}) \) and set \( \alpha = \mu(\tilde{a}) \). Then \( G \times_{\sigma} (\tilde{a})^{+} \) is isomorphic to \( (\alpha)^{+} \) as \( \Pi(\tilde{Y}) \)-complexes.

Proof. We start with two definitions:

\[
(\tilde{a})^{+} = \tilde{G}/\tilde{\rho}(\tilde{a}) \sqcup \{\ast\}, \quad \text{and} \quad (\alpha)^{+}_{\tilde{\beta}} = \{\tilde{g}(\tilde{a}) \mid \tilde{g} \in \tilde{G}, \tilde{g} \tilde{a} \leq \tilde{\beta} \} \sqcup \{\ast\} \quad \text{for} \quad \tilde{\beta} \in \Pi(\tilde{Y}).
\]

Set \( \tilde{X} = (\tilde{a})^{+} \) and \( X = G \times_{\sigma} (\tilde{a})^{+} = G \times_{\sigma} \tilde{X} \). First we investigate the cardinality of \( \tilde{X} \) and \( X \) respectively.

It is obvious that \( |\tilde{X}| = |\tilde{G}/\tilde{\rho}(\tilde{a})| + 1 \), where \( |\tilde{X}| \) is the the cardinality of \( \tilde{X} \). Notice that

\[
|X| = |\tilde{G}/\pi(\tilde{a})| + 1
= |(G/\rho(\alpha))| + 1
= |(\alpha)^{+}|.
\]

Next we shall define a map \( f : X \to (\alpha)^{+} \) given by \([1, \tilde{g}(\tilde{a})] \mapsto (\tilde{g}) \rho(\alpha) \), where the basepoint is mapped to the basepoint. This map is well-defined, \( \sigma \) being surjective, with the result that \( f \) is surjective. Since
$|X|$ equals $|(\alpha)^{+}|$, $f$ is also injective. In the following we shall verify that $f$ is a $G$-map. Choose $\tilde{\alpha} \in \sigma^{-1}(\alpha)$ for any $\alpha \in G$. Then

$$
\begin{align*}
 f(a[1, \tilde{g}\rho(\tilde{\alpha})]) &= f([a, \tilde{g}\rho(\tilde{\alpha})]) \\
 &= f([\sigma(\tilde{a}), \tilde{g}\rho(\tilde{\alpha})]) \\
 &= f([1, a\tilde{g}\rho(\tilde{\alpha})]) \\
 &= \sigma(a\tilde{g})\rho(\alpha) \\
 &= \sigma(\tilde{a})\sigma(\tilde{g})\rho(\alpha) \\
 &= af([1, \tilde{g}\rho(\tilde{\alpha})]).
\end{align*}
$$

Thus $f$ is a $G$-CW-complex isomorphism. It remains to prove that $f$ is a $\Pi(Y)$-map. Remark that the basepoint of $X$ is mapped to the basepoint of $\tilde{X}$ by $f$. For $x \in X_\beta \setminus \{*\}$, it suffices to verify that $f(x) \in (\alpha)_\beta^+$ for any $\beta \in \Pi(Y)$. Let $\beta^+$ be an element of $\mu^{-1}(\beta)$. Since $p_X : \tilde{X} \rightarrow X$ is surjective and $X_\beta = p_X(\tilde{X}_\beta)$, there exists $\tilde{x} \in \tilde{X}_\beta$ such that $x = p_X(\tilde{x}) = [1, \tilde{x}]$. By the definition of $\tilde{X}_\beta = (\tilde{\alpha})_{\beta}^+$, the point $\tilde{x}$ is written in the form: $\tilde{x} = \tilde{g}_0\rho(\tilde{\alpha})$ with $\tilde{g}_0\alpha \leq \beta$, where $\tilde{g}_0$ is a certain element of $G$. The following holds:

$$
\begin{align*}
 f(x) &= f([1, \tilde{x}]) \\
 &= f([1, \tilde{g}_0\rho(\tilde{\alpha})]) \\
 &= \sigma(\tilde{g}_0)\rho(\alpha) \quad \text{with} \quad \sigma(\tilde{g}_0)\mu(\tilde{\alpha}) \leq \mu(\beta^+).
\end{align*}
$$

Hence we have $f(x)$ lies in

$$(\alpha)_\beta^+ = \{g\rho(\alpha) \mid g \in G, \quad g\alpha \leq \beta\} \Pi \{*, \},$$

which asserts $f$ is a $\Pi(Y)$-map. It follows at once that $f$ is an isomorphism between $\Pi(Y)$-complexes. 

For each $\alpha \in \Pi(Y)$, take $a \in g^{-1}(\alpha)$. Suppose that $[\tilde{x}] = [\tilde{z}]$. Then $\tilde{\chi}(\tilde{X}) = \tilde{\chi}(\tilde{Z})$ for all $\tilde{\chi} \in \Pi(\tilde{Y})$. We have already seen

$$
\begin{align*}
 (G \times_\sigma \tilde{X})_\alpha &= p_X(\tilde{X}_\alpha), \quad \text{and} \\
 (G \times_\sigma \tilde{Z})_\alpha &= p_X(\tilde{Z}_\alpha).
\end{align*}
$$

Now,

$$
\tilde{\chi}(p_X(\tilde{X}_\alpha)) = \tilde{\chi}(\tilde{X}_\alpha)/|\pi| = \tilde{\chi}(\tilde{Z}_\alpha)/|\pi| = \tilde{\chi}(p_X(\tilde{Z}_\alpha)).
$$

Hence we have $\tilde{\chi}((G \times_\sigma \tilde{X})_\alpha) = \tilde{\chi}((G \times_\sigma \tilde{Z})_\alpha)$ for all $\alpha \in \Pi(Y)$, which means $[G \times_\sigma \tilde{X}] = [G \times_\sigma \tilde{Z}]$. Thus the canonical correspondence $[\tilde{x}] \mapsto [G \times_\sigma \tilde{x}]$ gives a well-defined map $\Omega(\tilde{G}, \Pi(\tilde{Y})) \rightarrow \Omega(G, \Pi(Y))$ and it has been denoted by $\nu$.

**Theorem 3.12.** ([4, Proposition 3.5]) The map $\nu$ is an isomorphism.

**Proof.** For two elements $[\tilde{x}_1], [\tilde{x}_2] \in \Omega(\tilde{G}, \Pi(\tilde{Y}))$, it is easily verified that

$$
p_X(\tilde{x}_1 \vee \tilde{x}_2) = p_X(\tilde{x}_1) \vee p_X(\tilde{x}_2).
$$

Then we have the following:

$$
\begin{align*}
 \nu([\tilde{x}_1] + [\tilde{x}_2]) &= \nu([\tilde{x}_1] \vee [\tilde{x}_2]) \\
 &= [G \times_\sigma (\tilde{x}_1 \vee \tilde{x}_2)] \\
 &= [G \times_\sigma \tilde{x}_1] + [G \times_\sigma \tilde{x}_2] \\
 &= \nu([\tilde{x}_1]) + \nu([\tilde{x}_2]).
\end{align*}
$$

Thus $\nu$ is a homomorphism. By 2.2,

$$
\Omega(G, \Pi(Y)) = \bigoplus_{\alpha} [(\alpha)^+]_X.
$$
where \([\alpha]\) runs over \(\Pi(Y)/G\), hence by Proposition 3.2 and Lemma 3.11, \(\nu\) is surjective. We can write
\[
[X_1] = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^1 (\tilde{\alpha})^+, \quad \text{and}
[X_2] = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^2 (\tilde{\alpha})^+,
\]
where \(n_{\tilde{\alpha}}^1, n_{\tilde{\alpha}}^2 \in \mathbb{Z}\). By Lemma 3.11, it holds that
\[
\nu([X_1]) = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^1 [G \times_\sigma (\tilde{\alpha})^+] = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^1 (\mu(\tilde{\alpha}))^+, \quad \text{and}
\nu([X_2]) = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^2 [G \times_\sigma (\tilde{\alpha})^+] = \sum_{\tilde{\alpha} \in \Pi(Y)/\tilde{G}} n_{\tilde{\alpha}}^2 (\mu(\tilde{\alpha}))^+.
\]
Note that \(\{\mu((\tilde{\alpha})^+) | \tilde{\alpha} \in \Pi(Y)/\tilde{G}\}\) is a basis of \(\Omega(G, \Pi(Y))\) by Proposition 3.5. Thus \(\nu([X_1]) = \nu([X_2])\) implies that each of the coefficients is equal, hence only if \([X_1] = [X_2]\). This shows that \(\nu\) is injective, and therefore an isomorphism. \(\square\)

**Proposition 3.13.** The set \(\nu(\Phi(\tilde{G}, \Pi(Y)))\) is contained in \(\Phi(G, \Pi(Y))\).

**Proof.** Let \(x \in \Phi(\tilde{G}, \Pi(Y))\). Then \(x\) is represented by \(\tilde{X}\) for some \(\Pi(Y)\)-resolution \(\tilde{X}\). Then \(\nu([\tilde{X}]) = [G \times_\sigma \tilde{X}^*]\). Since \(\tilde{X}(\tilde{X}^*) = 0\),
\[
\tilde{X}(G \times_\sigma \tilde{X}^*) = \tilde{X}(\tilde{X}^*)/|\pi| = 0.
\]
For \(\alpha \in \Pi(Y)\) with \(\alpha \neq m\) (where \(m\) is a unique maximal element of \(\Pi(Y)\)),
\[
\tilde{X}((G \times_\sigma \tilde{X}^*)_\alpha) = \tilde{X}(p_{\tilde{X}}(\tilde{X}^*_{\alpha})) \quad \text{(for an arbitrarily chosen} \ \tilde{\beta} \in \mu^{-1}(\alpha))
\]
\[= \tilde{X}(p_{\tilde{X}}(\tilde{X}^*_{\beta}))
\]
\[= \tilde{X}((G \times_\sigma \tilde{X})_\alpha).
\]
Since \(G \times_\sigma \tilde{X}\) is a \(\Pi(Y)\)-resolution, we have \(\nu(x) = \nu([\tilde{X}]) \in \Phi(G, \Pi(Y))\). \(\square\)

**4 PROOF OF THEOREM 1.2**

In the following, we shall first define groups \(\pi, G\) and \(\tilde{G}\), second define a finite \(\tilde{G}\)-CW-complex \(\tilde{Y}\) using the join operator \(*\), and finally check that \(\tilde{Y}\) is connected and simply connected, and that the \(G\)-CW-complex \(Y = \tilde{Y}/\pi\) satisfies \(\pi_1(Y) \cong \pi\) and \(\nu(\Phi(G, \Pi(Y)) \neq \Phi(G, \Pi(Y))\). Define
\[
\pi = \mathbb{Z}_p, \quad G = \mathbb{Z}_p \times \mathbb{Z}_q, \quad \text{and} \quad \tilde{G} = \pi \times G.
\]
Let \(\mathbb{Z}_p^\prime\) be a subgroup of \(\pi \times \mathbb{Z}_p\) of order \(p\) such that \(\mathbb{Z}_p^\prime \neq \pi \times \{1\}\) nor \(\{1\} \times \mathbb{Z}_p\). Next define
\[
B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +1) = (\tilde{G}/(\mathbb{Z}_p^\prime \times \mathbb{Z}_q) \ast \tilde{G}/(\mathbb{Z}_p^\prime \times \mathbb{Z}_q) \times \{1\}),
\]
\[
B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +2) = (\tilde{G}/(\mathbb{Z}_p^\prime \times \mathbb{Z}_q) \ast \tilde{G}/(\mathbb{Z}_p^\prime \times \mathbb{Z}_q) \times \{2\}),
\]
\[
B(\mathbb{Z}_p \times \mathbb{Z}_q, -1) = (\tilde{G}/(\mathbb{Z}_p \times \mathbb{Z}_q) \ast \tilde{G}/(\mathbb{Z}_p \times \mathbb{Z}_q) \times \{1\}),
\]
\[
B(\mathbb{Z}_p \times \mathbb{Z}_q, -2) = (\tilde{G}/(\mathbb{Z}_p \times \mathbb{Z}_q) \ast \tilde{G}/(\mathbb{Z}_p \times \mathbb{Z}_q) \times \{2\}),
\]
and
\[
B(\mathbb{Z}_p^\prime, +) = B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +1) \ast B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +2),
\]
\[
B(\mathbb{Z}_p, -) = B(\mathbb{Z}_p \times \mathbb{Z}_q, -1) \ast B(\mathbb{Z}_p \times \mathbb{Z}_q, -2),
\]
\[
B(\mathbb{Z}_q, 1) = B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +1) \ast B(\mathbb{Z}_p \times \mathbb{Z}_q, -1),
\]
\[
B(\mathbb{Z}_q, 2) = B(\mathbb{Z}_p^\prime \times \mathbb{Z}_q, +2) \ast B(\mathbb{Z}_p \times \mathbb{Z}_q, -2).
\]
Further set
\[\tilde{Y} = (B(\mathbb{Z}_p^\prime, +) \ast B(\mathbb{Z}_p, -) \ast B(\mathbb{Z}_q, 1) \ast B(\mathbb{Z}_q, 2)) \ast \tilde{G}.
\]
Then clearly $\tilde{Y}$ is a finite $\tilde{G}$-CW-complex, moreover connected and simply connected. Define $Y = \tilde{Y}/\pi$. Since $\pi$ acts freely on $\tilde{Y}$, $\pi_1(Y)$ is isomorphic to $\pi$.

In the remainder of this section, we shall prove that $\Phi(\tilde{G}, \tilde{\Pi}) = 0$ and $\Phi(G, \Pi) \neq 0$, where $\tilde{\Pi} = \Pi(\tilde{Y})$ and $\Pi = \Pi(Y)$, which concludes the proof of Theorem 1.2.

**Proposition 4.1.** The module $\Phi(\tilde{G}, \tilde{\Pi})$ is a trivial group.

**Proof.** It is easy to see that $\tilde{\Pi}$ consists of 9 elements, that is,

$$\tilde{\Pi} = \{\beta(\mathbb{Z}_p \times \mathbb{Z}_q, +1), \beta(\mathbb{Z}_p \times \mathbb{Z}_q, +2), \beta(\mathbb{Z}_p \times \mathbb{Z}_q, -1), \beta(\mathbb{Z}_p \times \mathbb{Z}_q, -2), \beta(\mathbb{Z}_p', +),$$

$$\beta(\mathbb{Z}_p, -), \beta(\mathbb{Z}_q, 1), \beta(\mathbb{Z}_q, 2), \tilde{m}\}$$

such that

$$|\beta(\mathbb{Z}_p', +)| = B(\mathbb{Z}_p', +), \quad \rho(\beta(\mathbb{Z}_p', +)) = \mathbb{Z}_p',$$

$$|\beta(\mathbb{Z}_p, -)| = B(\mathbb{Z}_p, -), \quad \rho(\beta(\mathbb{Z}_p, -)) = \mathbb{Z}_p,$$

$$|\beta(\mathbb{Z}_q, 1)| = B(\mathbb{Z}_q, 1), \quad \rho(\beta(\mathbb{Z}_q, 1)) = \mathbb{Z}_q,$$

$$|\beta(\mathbb{Z}_q, 2)| = B(\mathbb{Z}_q, 2), \quad \rho(\beta(\mathbb{Z}_q, 2)) = \mathbb{Z}_q,$$

$$|\tilde{m}| = \tilde{Y}, \quad \rho(\tilde{m}) = \{1\}.$$

The $\tilde{G}$-poset $\tilde{\Pi}$ is illustrated in Figure 2.

![Fig.2](image)

We recall

$$\mathcal{P}(\tilde{\Pi}) = \{\alpha \in \tilde{\Pi} \mid \rho(\alpha) \text{ is a subgroup of } \tilde{G} \text{ of prime power order}\}, \quad \text{and}$$

$$\mathcal{S}(\tilde{G}, \alpha) = \{K \in \mathcal{S}(\tilde{G}) \mid \rho(\alpha) \cdot K \subseteq \tilde{G}_\alpha \quad \text{and} \quad K/\rho(\alpha) \text{ is cyclic}\}.$$
We set \( \tilde{\mathcal{K}} = \{(\alpha, K) \mid \alpha \in \mathcal{P}(), K \in S(\tilde{G}, \alpha)\} \). Then, define the homomorphism
\[
\bar{\chi}(\alpha, K) : \Gamma (\tilde{G}, \tilde{\Pi}) \to \mathbb{Z}
\]
by \( \bar{\chi}(\alpha, K)([Z]) = \bar{\chi}(Z^K) \) for \( [Z] \in \Gamma (\tilde{G}, \tilde{\Pi}) \) and \((\alpha, K) \in \tilde{\mathcal{K}}\), and the homomorphism
\[
\bar{\chi}_\alpha : \Gamma (\tilde{G}, \tilde{\Pi}) \to \mathbb{Z}
\]
by \( \bar{\chi}_\alpha([Z]) = \bar{\chi}(Z) \) for \( [Z] \in \Gamma (\tilde{G}, \tilde{\Pi}) \) and \( \alpha \in \tilde{\Pi} \).

Since that \( \Phi(\tilde{G}, \tilde{\Pi}) = \{[Z] \in \Gamma (\tilde{G}, \tilde{\Pi}) \mid \bar{\chi}(Z^K) = 0, \text{ for all } \alpha \in \mathcal{P}(\tilde{\Pi}) \text{ and } K \in S(\tilde{G}, \alpha)\}\),

\[
\Phi(\tilde{G}, \tilde{\Pi}) = \ker \left[ \bigoplus_{(\alpha, K) \in \tilde{\mathcal{K}}} \bar{\chi}(\alpha, K) : \Gamma (\tilde{G}, \tilde{\Pi}) \to \bigoplus_{(\alpha, K) \in \tilde{\mathcal{K}}} \mathbb{Z} \right] \subset \ker \left[ \bigoplus_{(\alpha, K) \in \tilde{\mathcal{K}}} \bar{\chi}_\alpha : \Gamma (\tilde{G}, \tilde{\Pi}) \to \bigoplus_{(\alpha, K) \in \tilde{\mathcal{K}}} \mathbb{Z} \right]
\]

where \( \tilde{\mathcal{K}} := \{(\alpha, K) \in \tilde{\mathcal{K}} \mid \tilde{Y}_\alpha^K \text{ is connected}\} \). It suffices to prove that
\[
\ker(\bar{\chi}(\alpha, K)_{(\alpha, K) \in \tilde{\mathcal{K}}}) \subset \tilde{\mathcal{K}}
\]
is a trivial group. Since \( \tilde{Y}_\alpha^K \) is connected for \( (\alpha, K) \in \tilde{\mathcal{K}} \), we define \( \phi : \tilde{\mathcal{K}} \to \tilde{\Pi} \) by \( \phi(\alpha, K) = \text{the component of } \tilde{Y}_\alpha^K \). Furthermore, \( \tilde{Z}_\alpha^K = \tilde{Z}_{\phi(\alpha, K)} \) for \( (\alpha, K) \in \tilde{\mathcal{K}} \), and so we have \( \bar{\chi}(\alpha, K)([Z]) = \bar{\chi}_\phi(\alpha, K)([Z]) \). Remark that \( \phi(\tilde{\mathcal{K}}) = \tilde{\Pi} \). It follows at once that \( \ker(\bar{\chi}(\alpha, K)_{(\alpha, K) \in \tilde{\mathcal{K}}}) \) is a trivial group. \( \square \)

**Proposition 4.2.** The module \( \Phi(G, \Pi) \) is not a trivial group.

**Proof.** The G-poset \( \Pi = \Pi(Y) \) consists of 9 elements as follows:

\[
\begin{align*}
\Pi(Y) &= \prod_{H \in S(G)} \pi_0(Y^H) \\
&= \prod_{H \in S(G)} \pi_0((\tilde{Y}/\mathbb{Z}_p)^H) \\
&= \pi_0((\tilde{Y}/\mathbb{Z}_p)^{Z \times Z}) \prod_{H \in S(G)} \pi_0((\tilde{Y}/\mathbb{Z}_p)^Z) \prod_{H \in S(G)} \pi_0((\tilde{Y}/\mathbb{Z}_p)^{11}) \\
&= \{\mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, +)), \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, +2)), \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -)), \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -2))\} \prod\{\mu(\beta(\mathbb{Z}_p, 1)), \mu(\beta(\mathbb{Z}_p, 2))\} \prod\{\mu(\mathcal{m})\}
\end{align*}
\]

We write the elements of \( \Pi(Y) \) as follows: \( \alpha_1 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, +)), \alpha_2 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, +2)), \alpha_3 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -)), \alpha_4 := \mu(\beta(\mathbb{Z}_p \times \mathbb{Z}_q, -2)), \alpha_5 := \mu(\beta(\mathbb{Z}_p, +)), \alpha_6 := \mu(\beta(\mathbb{Z}_p, -)), \alpha_7 := \mu(\beta(\mathbb{Z}_q, 1)), \alpha_8 := \mu(\beta(\mathbb{Z}_q, 2)), m := \mu(\mathcal{m}) \).

It suffices to prove that \( \omega = [(\alpha_1)^+] + [(\alpha_4)^+] - [(\alpha_2)^+] - [(\alpha_3)^+] \) lies in \( \Omega(G, \Pi) \) and \( \omega \neq 0 \). However, by 2.5, it is clear that \( \omega \neq 0 \). Since \( G = \mathbb{Z}_p \times \mathbb{Z}_q \), we have that \( \mathcal{P}(\Pi) = \{m, \alpha_5, \alpha_6, \alpha_7, \alpha_8\} \). We must show that
\[
\bar{\chi}(X^K) = 0 \quad \text{for all } \alpha \in \mathcal{P}(\Pi) \text{ and } K \in S(G, \alpha),
\]
where \( X \) is a \( \Pi \)-complex representing \( \omega \).

Consider the case of \( \alpha = \alpha_5 \). Then, \( S(G, \alpha) = \{\mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_q\} \). For \( K = \mathbb{Z}_p \), the following hold:

\[
\begin{align*}
\bar{\chi}((\alpha_1)_{\mathbb{Z}_p}^{\mathbb{Z}_p}) &= \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \\
\bar{\chi}((\alpha_4)_{\mathbb{Z}_p}^{\mathbb{Z}_p}) &= \chi((\bullet)) = 0, \\
\bar{\chi}((\alpha_2)_{\mathbb{Z}_p}^{\mathbb{Z}_p}) &= \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \quad \text{and} \\
\bar{\chi}((\alpha_3)_{\mathbb{Z}_p}^{\mathbb{Z}_p}) &= \chi((\bullet)) = 0.
\end{align*}
\]
For $K = \mathbb{Z}_p \times \mathbb{Z}_q$, the following hold:

\[ \bar{\chi}(\alpha_1^{Z_p \times Z_q}) = \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \]
\[ \bar{\chi}(\alpha_4^{Z_p \times Z_q}) = \bar{\chi}({\ast}) = 0, \]
\[ \bar{\chi}(\alpha_2^{Z_p \times Z_q}) = \chi(G/(\mathbb{Z}_p \times \mathbb{Z}_q)) = 1, \text{ and} \]
\[ \bar{\chi}(\alpha_3^{Z_p \times Z_q}) = \bar{\chi}({\ast}) = 0. \]

Hence we obtain

\[ \bar{\chi}(X^K_\alpha) = 0. \]

By arguments similar to the above, we obtain

\[ \bar{\chi}(X^K_\alpha) = 0 \text{ for all } \alpha = \alpha_5, \alpha_7, \alpha_8, m, \text{ and } K \in S(G, \alpha). \]

Therefore $\omega$ lies in $\Phi(G, \Pi)$. 

\[ \square \]

**Remark 4.3.** Further computation proves that $\Phi(G, \Pi) \cong \mathbb{Z}$. 

**References**


