A numerical analysis of slow oscillations in dynamics of coupled systems

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Abstract

We study a system that models a problem in which an oscillatory unit is coupled to a passive medium. We analyze the case in which an RCL circuit is coupled to an RC circuit. Some numerical results indicate when slow oscillations occur in coupled systems.

KEYWORDS: coupled circuits, Van der Pol's equation, slow oscillations.

1. Introduction

We study a system that can generate an oscillatory dynamics with a large period out of a fast oscillator. Suppose that the state of some oscillatory unit is governed by the following n-dimensional system when the unit is isolated from a surrounding medium.

\[ \frac{dx_1}{dt} = f(x_1). \]  

Suppose also that the unit is coupled to a passive medium, and that the states of the unit and the medium are governed by the following system.

\[ \frac{dx_1}{dt} = f(x_1) + \delta P(x_0 - x_1), \]
\[ \frac{dx_0}{dt} = \epsilon \delta P(x_1 - x_0). \] 

Here \( x_0 \) represents the state of the medium and \( x_1 \) represents the state of the unit. \( \epsilon \) is defined by \( \epsilon = V_1/V_0 \), where \( V_1 \) is the volume of the unit and \( V_0 \) is the volume of the medium. \( P \) is an \( n \times n \) constant matrix of permeability coefficients, and \( \delta \) is the parameter which measures the coupling strength. A biological problem for which (2) is a model is found in [2]. We assume that the (1) has an orbitally asymptotically stable periodic solution \( x_1 = \eta(t) \) of least period \( T > 0 \). Periodic solutions of (2) in case the eigenvalues of \( P \) all have positive real parts are studied in [4] and [5]. Here we study the case \( P \) is singular. Specifically, we assume that

\[ P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \]

and that (1) is the following planar system.

\[ \kappa \frac{dv}{dt} = g(v, w), \]
\[ \frac{dw}{dt} = h(v, w). \]
Figure 1: Coupled electrical circuits. An RCL circuit has a resistor $R$, an inductor $L$, and a capacitor $C$. An RC circuit has a resistor $R_0$ and a capacitor $C_0$. The RCL circuit is coupled to the RC circuit.

Here we assume that $\kappa > 0$. Under the assumptions made in Section 2, (3) has an orbitally asymptotically periodic solution for all sufficiently small $\kappa > 0$. Now (2) becomes

$$
\begin{align*}
\kappa \frac{dv}{dt} &= g(v, w), \\
\frac{dw}{dt} &= h(v, w) + \delta p(w, x), \\
\frac{dx}{dt} &= -\delta p(w, x),
\end{align*}
$$

where $p(w, x) = (x - w)$. Coupled electrical circuits give rise to an example of such a system. Suppose that an RCL circuit is coupled to an RC circuit (cf. Figure 1), and that the uncoupled RCL circuit is modeled by the following system.

$$
\begin{align*}
L \frac{di_L}{dt} &= v_C - F(i_L), \\
C \frac{dv_C}{dt} &= -i_L.,
\end{align*}
$$

Here $i_L$ is the current through the inductor $L$ and $v_C$ is the voltage drop at the capacitor $C$. The characteristic of the resistor $R$ is the graph of $v_R = F(i_R)$ in the $(i_R, v_R)$ plane. Here $i_R$ is the current through the resistor $R$ and $v_R$ is the voltage drop at the resistor. We assume that $F(x) = x^3 - x$. $L$ is the inductance of the inductor $L$ and $C$ is the capacitance of the capacitor $C$. For $L = 1$ and for $C = 1$, (5) is Van der Pol's equation. The dynamics of the coupled circuits is governed by the following system.

$$
\begin{align*}
L \frac{di_L}{dt} &= v_C - f(i_L), \\
C \frac{dv_C}{dt} &= -i_L + \frac{1}{R_0} (v_{C_0} - v_C), \\
C_0 \frac{dv_{C_0}}{dt} &= -\frac{1}{R_0} (v_{C_0} - v_C).
\end{align*}
$$
Here $v_{C_0}$ is the voltage drop at the capacitor $C_0$, and $C_0$ is the capacitance of the capacitor. The characteristic of the resistor $R_0$ is $v_{R_0} = R_0 i_{R_0}$ in the $(i_{R_0}, v_{R_0})$ plane. Here $i_{R_0}$ is the current through the resistor $R_0$ and $v_{R_0}$ is the voltage drop at the resistor.

The substitutions $\tau = Ct$, $v = -i_L$, $w = v_C$, and $z = v_{C_0}$ in (5) and (6) lead to the following systems.

$$\begin{align*}
\kappa \frac{dv}{dt} &= -w - (v^3 - v), \\
\frac{dw}{dt} &= v, \\
\kappa \frac{dv}{dt} &= -w - (v^3 - v), \\
\frac{dw}{dt} &= v + \delta (x - w), \\
\frac{dx}{dt} &= -\epsilon \delta (x - w).
\end{align*}$$

(7)

(8)

When the functions $g(v, w)$, $h(v, w)$, and $p(w, x)$ are defined by

$$\begin{align*}
g(v, w) &= -w - (v^3 - v), \\
h(v, w) &= v, \\
p(w, x) &= x - w,
\end{align*}$$

and when the parameters $\kappa$, $\delta$, and $\epsilon$ are given by

$$\begin{align*}
\kappa &= \frac{L}{C}, \\
\delta &= \frac{1}{R_0}, \\
\epsilon &= \frac{C}{C_0},
\end{align*}$$

we find that (7) and (8) are special cases of (3) and (4), respectively. In Sections 2 and 3, we show some numerical results concerning (8).

2. Systems in the Singular Limit

We assume that the $v$-nullcline of (3) is given by a smooth function $w = \alpha(v)$, and that

$$\alpha'(v) = \begin{cases} < 0, & v < v_{t}, \\
> 0, & v_{t} < v < v_{r}, \\
< 0, & v > v_{r},
\end{cases}$$

$$g_w(v, \alpha(v)) < 0 \text{ for all } v \in \mathbb{R},$$

$$g_{vv}(v_{t}, w_{t}) \neq 0, \text{ } w_{t} = \alpha(v_{t}),$$

$$g_{vv}(v_{r}, w_{r}) \neq 0, \text{ } w_{r} = \alpha(v_{r}).$$

We also assume that $h_v(v, w) > 0$ for all $(v, w) \in \mathbb{R}^2$, and that the $w$-nullcline of (3) is a smooth curve which intersects the $v$-nullcline exactly at one point $(v_0, w_0)$ with $v_t < v_0 < v_r$. Note that (7) meets these requirements. Figure 2 shows the nullclines of (7). Under these assumptions, (3) has a periodic solution for sufficiently small $\kappa > 0$ [1]. Some examples of such periodic solutions are numerically generated, and shown in Figures 3 and 4.
Figure 2: The nullclines of the planar system. The nullclines of (7) are shown.

Figure 3: A closed orbit of the planar system. A periodic solution of (7) for $\kappa = 0.5$ is numerically generated, and its orbit is shown. The period $T$ of the periodic solution is given approximately by $T \approx 4.969327$. 
Figure 4: A closed orbit of the planar system. A periodic solution of (7) for $\kappa = 0.1$ is numerically generated, and its orbit is shown. The period $T$ of the periodic solution is given approximately by $T \approx 2.878519$.

Figure 5: The inverse functions of $w = \alpha(v)$. $w = \alpha(v)$ is $v$-nullcline of (7). $v = \zeta_L(w)$ and $v = \zeta_r(w)$ are defined on the intervals $(w_L, \infty)$ and $(-\infty, v_r)$, respectively.
We denote by \( v = \zeta_t(w) \) the inverse function of \( w = \alpha(v) \) defined on the interval \((w_t, \infty)\). and by \( v = \zeta_r(w) \) the inverse function of \( w = \alpha(v) \) defined on the interval \((\infty, v_r)\) (cf. Figure 5). For sufficiently small \( \kappa > 0 \), the solutions of (3) are approximated by the solutions in the limit \( \kappa \to 0 \). In the limit \( \kappa \to 0 \), the solutions are restricted either to the curve \( v = \zeta_t(w) \) or to the curve \( v = \zeta_r(w) \). The solutions in the curves \( v = \zeta_t(w) \) and \( v = \zeta_r(w) \) are governed by the following equations (9) and (10), respectively.

\[
\frac{dw}{dt} = h(\zeta_t(w), w),
\]

\[
\frac{dw}{dt} = h(\zeta_r(w), w).
\]

In particular, the system has a discontinuous periodic solution (cf. Figure 6). The period \( T \) of this periodic solution is given by

\[
T = \int_{w_t}^{w_r} \frac{dw}{h(\zeta_t(w), w)} + \int_{w_t}^{w_r} \frac{dw}{h(\zeta_r(w), w)}.
\]

In the limit \( \kappa \to 0 \), the solutions of (7) are restricted either to the region \( \Omega_t \) defined by

\[
\Omega_t = \{(v, w, x) | v = \zeta_t(w), w \geq w_t \}
\]

or to the region \( \Omega_r \) defined by

\[
\Omega_r = \{(v, w, x) | v = \zeta_r(w), w \geq w_t \}.
\]

In \( \Omega_t \), the dynamics is governed by the following system.

\[
\frac{dw}{dt} = h(\zeta_t(w), w) + \delta p(w, x),
\]

\[
\frac{dx}{dt} = -\epsilon \delta p(w, x).
\]

This system is valid in the region \( \Lambda_t \) defined by

\[
\Lambda_t = \{(w, x) | \geq w_t \}.
\]
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Figure 7: The region $\Lambda_t$. The nullclines of (11) which is obtained from (8) are shown.

Similarly, in $\Omega_r$, the dynamics is governed by the system

\[
\begin{align*}
\frac{dw}{dt} &= h(z_r(w), w) + \delta p(w, x), \\
\frac{dx}{dt} &= -e\delta p(w, x),
\end{align*}
\]

which is valid in the region $\Lambda_r$ defined by

$$\Lambda_r = \{(w, x) \mid w \geq w_r\}.$$  

Figures (7) and (8) show some examples of nullclines of (11) and (12), respectively. Figures 9 and 10 show some examples of trajectories of solutions of (11) and (12), respectively.

In the limit $K \to 0$, we define solutions of (4) using solutions of (11) and (12). We denote by $\phi(t, P_0)$ the solution of (4) with the initial value $\phi(0, P_0) = P_0$. We also denote by $\phi_r(t, Q_0) = (\gamma_r(t, Q_0), \sigma_r(t, Q_0))$ the solution of (11) with the initial value $\phi_r(0, Q_0) = Q_0$, and by $\tau_r(Q_0)$ the time it takes for $\phi_r(t, Q_0)$ to reach the lines $w = w_r$. Similarly, we denote by $\phi_r(t, Q_0) = (\gamma_r(t, Q_0), \sigma_r(t, Q_0))$ the solution of (12) with the initial value $\phi_r(0, Q_0) = Q_0$, and by $\tau_r(Q_0)$ the time it takes for $\phi_r(t, Q_0)$ to reach the lines $w = w_r$. Suppose that $P_0 = (w_0, w_0, x_0) = (\zeta(t, w_0), w_0, x_0) \in \Omega_t$ with $Q_0 = (w_0, x_0) \in \Lambda_t$. Then

$$\phi(t, P_0) = (\zeta(\gamma_r(t, Q_0)), \gamma_r(t, Q_0), \sigma_r(t, Q_0))$$

for $0 \leq t < \tau_r(Q_0)$. Suppose that $Q_1 = (w_t, x_1) = \phi_r(\tau_r(Q_0), Q_0)$. At $t = \tau_r(Q_0)$, $\phi(t, P_0)$ jumps to the point $(\zeta(t, w_t), w_t, x_1)$, and for $\tau_r(Q_0) \leq t < \tau_r(Q_0) + \tau_r(Q_1)$,

$$\phi(t, P_0) = (\zeta(\gamma_r(t - \tau_r(Q_0), Q_1)), \gamma_r(t - \tau_r(Q_0), Q_1), \sigma_r(t - \tau_r(Q_0), Q_1)).$$

Suppose that $Q_2 = (w_r, x_2) = \phi_r(\tau_r(Q_1), Q_1)$. At $t = \tau_r(Q_0) + \tau_r(Q_1)$, $\phi(t, P_0)$ jumps to the point $(\zeta(t, w_r), w_r, x_2)$. In particular, $\phi(t, P_0)$ is a periodic solution provided that $w_0 = w_r$ and that $x_0 = x_2$. Now we define the map $\Phi$ by $\Phi(x_0) = x_2$, where

$$\begin{align*}
Q_0 &= (w_r, x_0), \\
Q_1 &= (w_t, x_1) = \phi_r(\tau_r(Q_0), Q_0), \\
Q_2 &= (w_r, x_2) = \phi_r(\tau_r(Q_1), Q_1)
\end{align*}$$

(Cf. figure 11). It follows that a fixed point of $\Phi$ corresponds to a periodic solution of (4). Such a periodic
Figure 8: The region $\Lambda_r$. The nullclines of (12) which is obtained from (8) are shown.

Figure 9: Trajectories in $\Lambda_t$. Some trajectories of solutions of (11) which is obtained from (8) are shown.
Figure 10: Trajectories in $\Lambda_r$. Some trajectories of solutions of (12) which is obtained from (8) are shown.

Figure 11: The definition of the map $\Phi$. The map $\Phi$ is defined by $\Phi(x_0) = x_2$, where $Q_0 = (w_r, x_0)$, $Q_1 = (w_t, x_1) = \phi_t(\tau_t(Q_0), Q_0)$, $Q_2 = (w_r, x_2) = \phi_r(\tau_r(Q_1), Q_1)$. Here (11) and (12) are obtained from (8), and the map is numerically generated.
solution is discontinuous. When a discontinuous periodic solution in the singular limit exists, (4) can have at least one periodic solution for all sufficiently small $\kappa > 0$. Orbits of such periodic solutions are close to the orbit of the discontinuous periodic solution [1].

3. Periodic Solutions in the Singular Limit

In this section, we show that the map constructed in Section 2 always has a fixed point for a given pair $(\delta, \epsilon)$. Let $\bar{Q} = (w_t, \eta(w_t))$, let $\bar{Q}_0 = (w_r, \bar{x}_0) = \phi_r(\bar{Q})$, and let $\bar{Q}_1 = (w_r, \bar{x}_1) = (w_r, \eta(w_r))$. Then $[\bar{x}_0, \bar{x}_1]$ is an invariant interval of the map $\Phi$. Here (11) and (12) are obtained from (8).

Figure 12: The invariant interval of the map. $\bar{Q}$, $\bar{Q}_0$, and $\bar{Q}_1$ are defined by $\bar{Q} = (w_t, \eta(w_t))$, $\bar{Q}_0 = (w_r, \bar{x}_0) = \phi_r(\bar{Q})$, and $\bar{Q}_1 = (w_r, \bar{x}_1) = (w_r, \eta(w_r))$. Then $[\bar{x}_0, \bar{x}_1]$ is an invariant interval of the map $\Phi$. Here (11) and (12) are obtained from (8).
Figure 13: A periodic solution of the singular system. A periodic solution of (8) in the limit $\kappa \to 0$ is numerically generated. Here $\delta = 1$ and $\epsilon = 0.5$. The period $T$ of the periodic solution is given approximately by $T \approx 1.974587$.

Figure 14: A periodic solution of the singular system. A periodic solution of (8) in the limit $\kappa \to 0$ is numerically generated. Here $\delta = 2$ and $\epsilon = 0.5$. The period $T$ of the periodic solution is given approximately by $T \approx 3.084079$. 
Figure 15: The fixed points of the map. Here (11) and (12) are obtained from (8), and the fixed points of the map $\Phi$ are numerically generated for $\delta = 1$ and for some values of $\epsilon$ between 0.01 and 0.5.

Figure 16: The period of the periodic solutions in the singular limit. The period of the periodic solutions of (8) in the singular limit is numerically generated for $\delta = 1$ and for some values of $\epsilon$ between 0.01 and 0.5.
Figure 17: The fixed points of the map. Here (11) and (12) are obtained from (8), and the fixed points of the map $\Phi$ are numerically generated for $\delta = 2$ and for some values of $\epsilon$ between 0.01 and 0.5.

Figure 18: The period of the periodic solutions in the singular limit. The period of the periodic solutions of (8) in the singular limit is numerically generated for $\delta = 2$ and for some values of $\epsilon$ between 0.01 and 0.5.
REFERENCES


