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Derivations on Matrix Near-Ring

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Abstract

The existence of a derivation in a near-ring is not known. We construct derivations on 2×2 matrix near-ring in the sense of [MW].

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1. Introduction

Let R be a ring identity 1 and $M_2(R)$ the 2×2 matrix ring over R. All derivations in $M_2(R)$ are known and it is used to determine the structure of a ring which has a derivation with invertible values in [BHL]. On the other hand, the notion of a derivation is useful in the theory of near-rings, and several properties of near-rings with derivations were given by [BM], [H] and [W]. But it is not known that there exists a derivation on matrix near-ring $M_2(N)$, where N is a right near-ring.

In this note, using a similar way as in the case of the matrix ring $M_2(R)$ over R, we construct derivations on matrix near-ring $M_2(N)$.

2. Preliminaries

The notion of a near-ring and related things are seen in his book [P]. But it is not well known, so we begin to give a definition of a near-ring.

Definition 2.1. A set N with two binary operations "$+$" and "$\cdot$" is called a *near-ring* if the following properties hold:

(a) $(N, +)$ is a group (not necessarily abelian),

(b) $(N, \cdot)$ is a semigroup,

(c) $(a+b)n = an + bn$ for any a, b, $n \in N$ ("right distributive law").

In view of (c), we call more precisely a "right near-ring". The left distributive law is defined similarly and when this is the case, it is called a left near-ring. We also define some other notions for a right near-ring N. N is called *zero-symmetric* if $a \cdot 0 = 0$ for any $a \in N$, and the set $C(N) := \{x \in N \mid xa = ax$ for any $a \in N\}$ is called the *center* of N. A map $d : N \to N$ is said to be a *derivation* if

$$d(a+b) = d(a) + d(b), \qquad d(ab) = d(a)b + ad(b) \quad (a, b \in N).$$

This is different in [P, p. 232], but it is the same as the definition of their papers [BM], [H] and [W].

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In this note, we mean that a near-ring is a right near-ring and zero-symmetric. The notion of a near-ring is not well known, but there are many examples of near-rings.

Example 2.2. (1) Let R be a commutative ring with identity and $R[X]$ the set of all polynomials with coefficients in R. Then $R[X]$ is an additive group under the usual addition of polynomials. For $f(X)$, $g(X) \in R[X]$, we define

$$f(X) \circ g(X) = f(g(X)),$$

that is, $\circ$ is a substitution. Then $(R[X], +, \circ)$ is a near-ring and zero-symmertic.

(2) Let V be a vector space over a field k. We call a map $f : V \to V$ an *affine map* if f is the sum of a linear map and a constant map. Then the set $A_{ff}(V)$ of all affine maps is a near-ring and zero-symmertic with pointwise addition and composition:

$$(f+g)(v) = f(v) + g(v), \quad (f \circ g)(v) = f(g(v)) \quad (f, g \in A_{ff}(V), v \in V).$$

Now, we give the definition of a matrix near-ring according to J. D. P. Meldrum and A. P. J. van der Walt [MW]. Let N be a near-ring and N^n the direct sum of n copies of the group $(N, +)$. Then N^n is also a near-ring as usual way. We denote $M(N^n)$ the set of all maps from $(N^n, +)$ into itself. Then $M(N^n)$ is a near-ring with pointwise addition and composition.

Let $\varepsilon_j = (0, ..., 0, 1, 0, ..., 0)$ (jth position is 1 and the other positions are zero). We define the following maps:

$$\iota_j : N \ni a \to a\varepsilon_j = (0, ..., 0, a, 0, ..., 0) \in N, \quad (a \text{ is the } j\text{th position})$$
$$\pi_j : N^n \to N, \quad \text{where} \quad \pi_j(\varepsilon_j) = 1, \quad \pi_j(\varepsilon_i) = 0 \ (i \neq j)$$
$$f^a : N \ni x \to ax \in N$$
$$f^a_{ij} = \iota_i f^a \pi_j : N^n \to N^n$$

Definition 2.3. ([MW, Definition 2.1]) The *near-ring of* $n \times n$ *matrices* $\mathbf{M}_n(N)$ over N is the subnear-ring of $M(N^n)$ generated by the set $\{f^a_{ij} \mid a \in N, 1 \leq i, j \leq n\}$.

Then $\mathbf{M}_n(N)$ is a (right) near-ring with identity ([MW, Proposition 2.2]). In their paper [BHL], Bergen, Herstein and Lanski gave all derivations on 2×2 matrix ring over R. Using this method, we try to construct derivations on 2×2 matrix near-ring $\mathbf{M}_2(N)$. By definition of f^x_{ij}, we see

$$f^x_{11} : N^2 \ni (a, b) \to (xa, 0) \in N^2, \quad f^x_{12} : N^2 \ni (a, b) \to (xb, 0) \in N^2$$
$$f^x_{21} : N^2 \ni (a, b) \to (0, xa) \in N^2, \quad f^x_{22} : N^2 \ni (a, b) \to (0, xb) \in N^2.$$

Thus the multiplications of f^x_{ij} with each other are similar to the multiplication of the matrix units in ring theory.

Lemma 2.4. *The multiplication of* f^x_{ij} *are follows:*

$$f^a_{ij} f^b_{j\ell} = f^{ab}_{i\ell}, \quad f^a_{ij} f^b_{k\ell} = 0, \quad (j \neq k)$$

for any $1 \leq i, j, k, \ell \leq 2$ *and* $a, b \in N$.

3. Construction of derivations

Let $a, b \in N$ and $x, y, z, w \in N$. We define

$$(f_{11}^x)' = f_{12}^{ax} + f_{21}^{bx}, \quad (f_{12}^y)' = -f_{11}^{by} + f_{22}^{by}, \quad (f_{21}^z)' = -f_{11}^{az} + f_{22}^{az}, \quad (f_{22}^w)' = -f_{12}^{aw} + f_{21}^{bw}.$$

Then we have

Lemma 3.1. *The following relations hold for any* $1 \leq i, j, k, \ell \leq 2$.

$(f_{ij}^x f_{j\ell}^y)' = (f_{i\ell}^{xy})' = (f_{ij}^x)' f_{j\ell}^y + f_{ij}^x (f_{j\ell}^y)'$,

$(f_{ij}^x f_{k\ell}^y)' = 0 = (f_{ij}^x)' f_{k\ell}^y + f_{ij}^x (f_{k\ell}^y)' \quad (j \neq k)$.

Proof. We only prove the case $i = j = k = \ell = 1$, because the other relations are proved by the similar way as in the first case. By definition, we have

$$(f_{11}^x)' f_{11}^y + f_{11}^x (f_{11}^y)' = (f_{12}^{ax} + f_{21}^{bx}) f_{11}^y + f_{11}^x (f_{12}^{ay} + f_{21}^{by}).$$

Since $M_2(N)$ is a right near-ring, then the first part is expanded and the second part is

$$f_{11}^x (f_{12}^{ay} + f_{21}^{by})(s,t) = f_{11}^x((ayt, 0) + (0, bys)) = f_{11}^x(ayt, bys) = (xayt, 0) = f_{12}^{xay}(s,t).$$

Thus $(f_{11}^x)' f_{11}^y + f_{11}^x (f_{11}^y)' = f_{21}^{bxy} + f_{12}^{xay}$. In this case, we see $f_{21}^{bxy} + f_{12}^{xay} = f_{12}^{xay} + f_{21}^{bxy}$. This prove the case $i = j = k = \ell = 1$.

Using this lemma, we have the following

Theorem 3.2. *There exists a derivation on the matrix near-ring* $M_2(N)$.

Proof. Applying the relations in Lemma 3.1 to $(a, b) \in N^2$, we see

$$(f_{ij}^x)' f_{k\ell}^y + f_{ij}^x (f_{k\ell}^y)' = f_{ij}^x (f_{k\ell}^y)' + (f_{ij}^x)' f_{k\ell}^y$$

for any $1 \leq i, j, k, \ell \leq 2$ and $x, y \in N$. Since $M_2(N)$ is generated by the set $\{f_{11}^x, f_{12}^y, f_{21}^z, f_{22}^w\}$ and using the above relation, we can easily see that the map

$$d : M_2(N) \ni f \to f' \in M_2(N)$$

is a derivation.

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