Computation of the Grothendieck-Witt ring of nonsingular Hermitian forms with positioning map

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The generalized Grothendieck-Witt ring of nonsingular Hermitian forms with positioning map is the cartesian product of the ordinary Grothendieck-Witt ring of nonsingular Hermitian forms and a map ring. We study the generalized Grothendieck-Witt ring by computing the map ring.

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1 Introduction

Throughout this paper, let $G$ be a finite group, $R$ a commutative ring with $1$, and $\Theta$ a finite $G$-set. The ordinary Grothendieck-Witt ring $GW_0(R,G)$ in A. Dress [10] was generalized to $GW_0(R,G,\Theta)$ in [5, p. 7], [13, p. 506], [16, p. 2356], [7, pp. 59-61], and the classical one is obtained as the case $\Theta = \emptyset$, namely $GW_0(R, G) = GW_0(R, G, \emptyset)$. Let $\mathfrak{S}_2$ denote the symmetric group of degree 2 and $r$ the generator of $\mathfrak{S}_2$. The cartesian product $\Theta \times \Theta$ is afforded the $G \times \mathfrak{S}_2$-action such that $g(t, t') = (gt, gt')$ and $\tau(t, t') = (t', t)$ for $g \in G$ and $t, t' \in \Theta$. Let $\text{Map}_G(\Theta \times \Theta, R)$ (resp. $\text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R)$) denote the set of all $G$-invariant (resp. $G \times \mathfrak{S}_2$-invariant) maps from $\Theta \times \Theta$ to $R$. This set $\text{Map}_G(\Theta \times \Theta, R)$ (resp. $\text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R)$) has a canonical ring structure and isomorphic to $\text{Map}(G/(\Theta \times \Theta), R)$ (resp. $\text{Map}((G \times \mathfrak{S}_2)/(\Theta \times \Theta), R)$), where $G/(\Theta \times \Theta)$ (resp. $(G \times \mathfrak{S}_2)/(\Theta \times \Theta)$) is the $G$-orbit space (resp. $G \times \mathfrak{S}_2$-orbit space) of $\Theta \times \Theta$. We define

$$M(G, \Theta, R) = \text{Map}_{G \times \mathfrak{S}_2}(\Theta \times \Theta, R).$$

If $R$ is a principal ideal domain then by [7] $GW_0(R,G,\Theta)$ is isomorphic to the cartesian product of $GW_0(R,G)$ and $M(G, \Theta, R)$ as rings. Let $RO(G)$ denote the real representation ring of $G$.

Theorem 1.1. The abelian group $Z[\frac{1}{2}] \otimes GW_0(Z, G, \Theta)$ is isomorphic to

$$Z[\frac{1}{2}] \otimes RO(G) \otimes M(G, \Theta, Z[\frac{1}{2}]).$$

The $Z$-rank of $GW_0(Z, G, \Theta)$ is the sum of the number of isomorphism classes of finite dimensional irreducible real $G$-modules and the $Z$-rank of $M(G, \Theta, Z)$. Moreover, the torsion subgroup of $GW_0(Z, G, \Theta)$ is annihilated by 4.

Theorem 1.2. Let $\Theta$ be a finite $G$-set isomorphic to $\prod_{i=1}^{t} \bigg( \coprod_{j=1}^{a_i} G/H_i \bigg)$ ($a_i \in \mathbb{N}$). Then $M(G, \Theta, R)$ is an $R$-free module of rank

$$\sum_{i=1}^{t} \left( a_i \right) \left| (G \times \mathfrak{S}_2) \backslash (G/H_i \times G/H_i) \right| + \frac{a_i^2 - a_i}{2} \left| G \backslash (G/H_i \times G/H_i) \right|$$

$$+ \sum_{1 \leq i < j \leq t} a_ia_j \left| G \backslash (G/H_i \times G/H_j) \right|,$$

where $\left| (G \times \mathfrak{S}_2) \backslash (G/H_i \times G/H_i) \right|$ stands for the number of $G \times \mathfrak{S}_2$-orbits in $G/H_i \times G/H_i$, and $\left| G \backslash (G/H_i \times G/H_j) \right|$ stands for the number of $G$-orbits in $G/H_i \times G/H_j$.

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As applications of these theorems, we compute the $\mathbb{Z}$-rank of $GW_0(\mathbb{Z}, G, \Theta)$ in the cases $G = A_5$, the alternating group of degree 5, and $SL(2, \mathbb{Z}_5)$, the special linear group of degree 2 with coefficients in $\mathbb{Z}_5$. The results are described in Sections 2 and 3.

2 The rank of for $GW_0(\mathbb{Z}, G, \Theta)$ for $G = A_5$

A complete set $\mathcal{R}_{A_5}$ of representatives for conjugacy classes of subgroups of $G = A_5$ is

$$\mathcal{R}_{A_5} = \{\{e\}, C_2, C_3, D_4, C_5, D_6, D_{10}, A_4, A_5\},$$

where $\{e\}$ is the unit group, $C_n$ ($n = 2, 3, 5$) are the cyclic groups of order $n$, generated by the elements $\sigma_n$:

$$\sigma_2 = (1, 2)(3, 4), \quad \sigma_3 = (1, 2, 3), \quad \sigma_5 = (1, 2, 3, 4, 5),$$

$D_{2n}$ are the dihedral groups of order $2n$ obtained by $D_{2n} = N_G(C_n)$, and $A_4$ is the alternating group on four letters $\{1, 2, 3, 4\}$.

![Figure 1: Conjugacy Class Subgroups of $A_5$](image)

**Lemma 2.1.** In the case $G = A_5$, $RO(G)$ is a free $\mathbb{Z}$-module of rank 5.

*Proof.* This follows from [8, Theorem 38.1] and Frobenius-Shur's Theorem, since $A_5 = SL(2, \mathbb{Z}_5)/C_2$. □

Let $\Theta$ be a finite $G$-set. A $G \times S_2$-orbit $X$ in $\Theta \times \Theta$ is called $S_2$-effective if $X$ contains precisely two $G$-orbits, and $S_2$-indecomposable if $X$ contains only one $G$-orbit.

The Program 5.6 for $G = A_5$ given in Section 5 provides the next result which was theoretically computed by M. Kubo [12].

**Lemma 2.2.** Let $G = A_5$ and $\mathcal{R}_{A_5} = \{H_i \mid i = 1, 2, \ldots, 9\}$ the complete set above of representatives for conjugacy classes of subgroups of $A_5$. Then the number of $G$-orbits in $G/H_i \times G/H_j$ is given as in Table 1 and the number of $G \times S_2$-orbits in $G/H_i \times G/H_i$ is given as in Table 2.

The notation in the tables reads that, for example in the case

$$2_{20 \times 1} + 4_{60 \times 1} + 1_{60 \times 2},$$

there exist two $S_2$-indecomposable $G \times S_2$-orbits of orbit-length 20, four $S_2$-indecomposable $G \times S_2$-orbits of orbit-length 60 and one $S_2$-decomposable $G \times S_2$-orbit of orbit-length 120.

The next theorem immediately follows from Theorem 2.2.
Table 1: The number of $G$-orbits in $G/H_i \times G/H_j$ where $G = A_5$

<table>
<thead>
<tr>
<th>$H_i$</th>
<th>orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>${e}$</td>
<td>$16_{60 \times 1} + 22_{60 \times 2}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$2_{30 \times 4} + 3_{60 \times 1} + 4_{60 \times 2}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$2_{20 \times 1} + 4_{60 \times 1} + 1_{60 \times 2}$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$1_{15 \times 1} + 1_{15 \times 2} + 3_{60 \times 1}$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$2_{12 \times 1} + 2_{60 \times 1}$</td>
</tr>
<tr>
<td>$D_6$</td>
<td>$1_{10 \times 1} + 1_{30 \times 1} + 1_{60 \times 1}$</td>
</tr>
<tr>
<td>$D_{10}$</td>
<td>$1_{6 \times 1} + 1_{30 \times 1}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$1_{5 \times 1} + 1_{20 \times 1}$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$1_{1 \times 1}$</td>
</tr>
</tbody>
</table>

Table 2: The number of $G \times S_2$-orbits in $G/H_i \times G/H_i$ where $G = A_5$

<table>
<thead>
<tr>
<th>${e}$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$D_{10}$</th>
<th>$A_4$</th>
<th>$A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>38</td>
<td>12</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Theorem 2.3. Let $G$ and $R_{A_5} = \{H_i \ | \ i = 1, 2, \cdots, 9\}$ be as above. Then the numbers $|(G \times S_2) \setminus (G/H_i \times G/H_i)|$ and $|G \setminus (G/H_i \times G/H_i)|$ appearing in Theorem 1.2 are given as in Tables 3 and 4, respectively.

Table 3: $|(G \times S_2) \setminus (G/H_i \times G/H_i)|$ where $G = A_5$

<table>
<thead>
<tr>
<th>${e}$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$D_4$</th>
<th>$C_5$</th>
<th>$D_5$</th>
<th>$D_{10}$</th>
<th>$A_4$</th>
<th>$A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>30</td>
<td>20</td>
<td>15</td>
<td>12</td>
<td>10</td>
<td>6</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>16</td>
<td>10</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$C_3$</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$D_4$</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_5$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_6$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_{10}$</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_4$</td>
<td></td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_5$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: $|G \setminus (G/H_i \times G/H_j)|$ where $G = A_5$
3 The rank of for $GW_0(Z, G, \Theta)$ for $G = SL(2, Z_5)$

We recall basic properties of $SL(2, Z_5)$, the special linear group of degree 2 on $Z_5$. A complete set $\mathcal{R}_{SL(2, Z_5)}$ of representatives for conjugacy classes of subgroups of $SL(2, Z_5)$ is given by

$$\mathcal{R}_{SL(2, Z_5)} = \{\{e\}, C_2, C_3, C_4, C_5, C_6, Q_8, C_{10}, Q_{12}, Q_{20}, \pi^{-1}(A_4), SL(2, Z_5)\},$$

where $\{e\}$ is unit group, $C_n$ ($n = 2, 3, 4, 5, 6, 10$) are the cyclic groups of order $n$, $Q_{4m}$ ($n = 2, 3, 5$) are the quaternion groups of order $4m$, namely

$$Q_{4m} = \langle a, b \mid b^m = a^2, b^{2m} = e, a^{-1}ba = b^{-1} \rangle,$$

and

$$\pi : SL(2, Z_5) \to SL(2, Z_5)/C_2 (= A_5); \ a \to aC_2.$$

![Figure 2: Conjugacy Class Subgroups of SL(2, Z_5)](image)

**Lemma 3.1.** In the case $G = SL(2, Z_5)$, $RO(G)$ is a free $Z$-module of rank 9.

**Proof.** This follows from [8, Theorem 38.1] and Frobenius-Shur's Theorem. □

**Lemma 3.2.** Let $G = SL(2, Z_5)$ and $\mathcal{R}_{SL(2, Z_5)} = \{H_i \mid i = 1, 2, \cdots, 12\}$ the complete set above of representatives for conjugacy classes of subgroups of $SL(2, Z_5)$. Then the number of $G$-orbits in $G/H_i \times G/H_j$ is given as in Table 5 and the number of $G \times \Theta_2$-orbits in $G/H_i \times G/H_i$ is given as in Table 6.

The next theorem immediately follows from Theorem 3.2.

**Theorem 3.3.** Let $G$ and $\mathcal{R}_{SL(2, Z_5)} = \{H_i \mid i = 1, 2, \cdots, 12\}$ be as above. Then the numbers $\left|\langle G \times \Theta_2 \rangle \backslash \langle G/H_i \times G/H_j \rangle\right|$ and $\left|\langle G \rangle \backslash \langle G/H_i \times G/H_j \rangle\right|$ appearing in Theorem 1.2 are given as in Tables 7 and 8, respectively.

4 Proofs of Theorems 1.1 and 1.2

The ordinary Grothendieck-Witt ring $GW_0(Z, G)$ is defined in [9, p. 742], and the Grothendieck-Witt ring $GW_0(R, G, \Theta)$ is defined in [5], [13], [16, §4] and [7, pp. 59–61].

**Proof of Theorem 1.1.** If $R$ is a principal ideal domain then by Theorem 1 of [7]

$$GW_0(R, G, \Theta) \cong GW_0(R, G) \oplus M(G, \Theta, R).$$

By definition, $M(G, \Theta, R) = \text{Map}_{G \times \Theta_1}(\Theta \times \Theta, R)$ is a free $R$-module. Thus Theorem 1.1 follows from [9, Theorem 3]. □
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
{\psi} & C_2 & C_3 & C_4 & C_5 & C_6 \\
\hline
{\psi} & 120_{120} & 60_{120} & 40_{120} & 30_{120} & 24_{120} & 20_{120} \\
C_2 & 60_{60} & 20_{120} & 30_{60} & 12_{120} & 20_{60} & 0 \\
C_3 & 4_{60} + 12_{120} & 10_{120} & 8_{120} & 24_{120} & 4_{120} & 120 \\
C_4 & 2_{30} + 14_{60} & 6_{120} & 10_{60} & 0 & 0 & 0 \\
C_5 & 4_{24} + 4_{120} & 4_{120} & 120 & 0 & 0 & 0 \\
C_6 & 2_{20} + 6_{60} & 20_{120} & 0 & 0 & 0 & 0 \\
Q_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{20} & 0 & 0 & 0 & 0 & 0 & 0 \\
\pi^{-1}(A_4) & 0 & 0 & 0 & 0 & 0 & 0 \\
SL(2, \mathbb{Z}_6) & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{The number of G-orbits in $G/H_i \times G/H_j$ where $G = \text{SL}(2, \mathbb{Z}_6)$}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Q_8 & C_{10} & Q_{12} & Q_{20} & \pi^{-1}(A_4) & SL(2, \mathbb{Z}_6) \\
\hline
15_{120} & 12_{120} & 10_{120} & 6_{120} & 5_{120} & 1_{120} \\
15_{60} & 12_{60} & 10_{60} & 6_{60} & 5_{60} & 1_{60} \\
5_{120} & 4_{120} & 1_40 + 3_{120} & 2_{120} & 2_{40} + 1_{120} & 1_{40} \\
3_{30} + 6_{60} & 6_{60} & 2_{30} + 4_{60} & 2_{30} + 2_{60} & 1_{30} + 2_{60} & 1_{30} \\
3_{120} & 2_{24} + 2_{120} & 2_{120} & 1_{24} + 1_{120} & 1_{120} & 1_{120} \\
5_{60} & 4_{60} & 1_{30} + 3_{60} & 2_{60} & 2_{20} + 1_{60} & 1_{20} \\
3_{15} + 3_{60} & 3_{60} & 3_{30} + 1_{60} & 3_{30} & 1_{15} + 1_{60} & 1_{15} \\
2_{12} + 2_{60} & 2_{60} & 1_{12} + 1_{60} & 1_{60} & 1_{12} & 1_{12} \\
\hline
\end{tabular}
\caption{The number of $G \times \mathfrak{S}_2$-orbits in $G/H_i \times G/H_j$ where $G = \text{SL}(2, \mathbb{Z}_6)$}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\{\psi\} & C_2 & C_3 & C_4 & C_5 & Q_8 & C_{10} & Q_{12} & Q_{20} & \pi^{-1}(A_4) & SL(2, \mathbb{Z}_6) \\
\hline
61 & 38 & 9 & 12 & 5 & 7 & 5 & 3 & 2 & 2 & 1 \\
\hline
\end{tabular}
\caption{${\mid (G \times \mathfrak{S}_2)\backslash(G/H_i \times G/H_j)\mid}$ where $G = \text{SL}(2, \mathbb{Z}_6)$}
\end{table}
Table 8: $|G\backslash(G/H_i \times G/H_j)|$ where $G = \text{SL}(2, \mathbb{Z}_5)$

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & $\{e\}$ & $C_2$ & $C_3$ & $C_4$ & $C_5$ & $C_6$ & $Q_8$ & $Q_{12}$ & $Q_{20}$ & $\pi^{-1}(A_4)$ & $\text{SL}(2, \mathbb{Z}_5)$ \\
\hline
$\{e\}$ & 120 & 60 & 40 & 30 & 24 & 20 & 15 & 12 & 10 & 6 & 5 & 1 \\
$C_2$ & 60 & 30 & 12 & 20 & 15 & 12 & 10 & 6 & 5 & 1 & & \\
$C_3$ & 16 & 10 & 8 & 5 & 4 & 4 & 2 & 3 & 1 & & \\
$C_4$ & 16 & 6 & 10 & 9 & 6 & 4 & 3 & 1 & & \\
$C_5$ & 8 & 4 & 3 & 4 & 2 & 2 & 1 & 1 & & \\
$C_6$ & 8 & 4 & 3 & 4 & 2 & 2 & 1 & 1 & & \\
$Q_8$ & 6 & 3 & 4 & 3 & 2 & 1 & & & & & \\
$Q_{12}$ & 4 & 2 & 2 & 1 & 1 & & & & & & \\
$Q_{20}$ & 3 & 2 & 2 & 1 & & & & & & & \\
$\pi^{-1}(A_4)$ & 2 & 1 & & & & & & & & & \\
$\text{SL}(2, \mathbb{Z}_5)$ & 1 & & & & & & & & & & \\
\hline
\end{tabular}

Proof of Theorem 1.2. Since
\[
\text{Map}_{G \times S_2}((\Theta_1 \cup \Theta_2) \times (\Theta_1 \cup \Theta_2), R)
\cong \text{Map}_{G \times S_2}((\Theta_1 \times \Theta_1, R) \oplus \text{Map}_{G \times S_2}(\Theta_2 \times \Theta_2, R) \oplus \text{Map}_G(\Theta_1 \times \Theta_2, R),
\]
we obtain
\[
\text{Map}_{G \times S_2}(\bigoplus_{i=1}^t \Theta_i) \times (\bigoplus_{i=1}^t \Theta_i), R)
\cong \bigoplus_{i=1}^t \text{Map}_{G \times S_2}(\Theta_i \times \Theta_i, R) \oplus \bigoplus_{1 \leq i < j \leq t} \text{Map}_G(\Theta_i \times \Theta_j, R).
\]
The next Lemma follows from this.

Lemma 4.1. If $\Theta \cong \bigoplus_{i=1}^t \Theta_i (a_i \in \mathbb{N})$ then
\[
M(G, \Theta, R) \cong \bigoplus_{i=1}^t \bigoplus_{k=1}^{a_i} M(G, G/H_i, R) \oplus \bigoplus_{k=1}^{(a_i^2 - a_i)/2} \text{Map}_G(G/H_i \times G/H_i, R)
\cong \bigoplus_{1 \leq i < j \leq t} \bigoplus_{k=1}^{a_i \times a_j} \text{Map}_G(G/H_i \times G/H_j, R).
\]

Let $R_G = \{H_i \mid 1 \leq i \leq t\}$ be a complete set of representatives for conjugacy classes of subgroups of $G$. By the Lemma above, $\text{Map}_{G \times S_2}(\Theta \times \Theta)$ is determined by the numbers of elements in $G \setminus (G/H_i \times G/H_j)$ and $(G \times S_2) \setminus (G/H_i \times G/H_i)$, where $H_i$ and $H_j$ run over $R_G$. \hfill \square

5 Appendix: Algorithms and Programmings

In the Appendix, we give the algorithms and programmings for Lemma 2.2 and Lemma 3.2.

Lemma 5.1. Let $H$ and $K$ be subgroups of $G$. Then the following hold.

1. $G/H \times G/K$ has the $G$-orbit decomposition
   \[
   \bigsqcup_{h \in H \cap G/K} G \cdot (eH, gK).
   \]

2. Each $G$-orbit $G \cdot (H, gH)$ is transposed to the $G$-orbit $G \cdot (H, g^{-1}H)$ by the generator $\tau$ of $S_2$. 
(3) A G-orbit $G \cdot (H,gH)$ is itself a $G \times S_2$-orbit if and only if $HgH = Hg^{-1}H$.

This lemma is well known but we give the proof for the reader’s convenience.

**Proof.** (1) Each G-orbit in $G/H \times G/K$ has a representative of the form $(eH,gK)$. The points $(eH,aK)$ and $(eH,bK)$, where $a$ and $b \in G$, lie in a same G-orbit if and only if $H a K = H b K$. Thus we get

$$G/H \times G/K = \bigsqcup_{HgK \in H \setminus G/K} G \cdot (eH,gK).$$

(2) The claim follows from $r(eH,gK) = (gH,eK)$ and $G \cdot (eH,gK) = G \cdot (eH,g^{-1}H)$.

(3) Since $(G \times S_2) \cdot (eH,gH) = G \cdot (eH,gH) \cup G \cdot (eH,g^{-1}H),$

$$G \times S_2 \cdot (eH,gH) = G \cdot (eH,gH)$$

holds if and only if $G \cdot (eH,gH) = G \cdot (eH,g^{-1}H)$, in other words $HgH = Hg^{-1}H$. □

**Proposition 5.2.** One has the formulae:

1. $|G \setminus (G/H \times G/K)| = |H \setminus G/K|,$
2. $|(G \times S_2) \setminus (G/H \times G/H)| = \left| \left\{ H g H \in H \setminus G/H \mid HgH = Hg^{-1}H \right\} \right|$

$$+ \left| \left\{ H g H \in H \setminus G/H \mid HgH \neq Hg^{-1}H \right\} \right| / 2.$$

**Proof.** This immediately follows from Lemma 5.1. □

We use GAP (Groups, Algorithms, and Programming) to compute the number of orbits $|G \setminus (G/H_i \times G/H_j)|$ and $|(G \times S_2) \setminus (G/H_i \times G/H_j)|$ for $G = A_5$ and $SL(2, \mathbb{Z}_5)$ together with Proposition 5.2 (see Algorithm 5.3 and Program 5.4).

**Algorithm 5.3.** Let $G$ be a finite group, $\mathcal{R}_G = \{H_i \mid i = 1, 2, \ldots, t\}$ a complete set of representatives for conjugacy classes of subgroups of $G$. $M(G)$ will be the matrix $\left( |H_i \setminus G/H_j| \right)$ (namely $\left( |G \setminus (G/H_i \times G/H_j)| \right)$), and $L(G \times S_2)$ will be the list of $\left( |G \times S_2) \setminus (G/H_i \times G/H_j)| \right)$.

**Input:** $G$

**Output:** $M(G)$, $L(G \times S_2)$

$\mathcal{R}_G := \{H_i \mid 1 \leq i \leq t\}$

For $1 \leq i \leq t$ Do

For $1 \leq j \leq t$ Do

$M(G)_{ij} := |H_i \setminus G/H_j|$

End Do

End Do

For $1 \leq i \leq t$ Do

$L(G \times S_2)_i := \left( |H_i g H_i \in H_i \setminus G/H_i \mid H_i g H_i = H_i g^{-1}H_i| \right)$

$$+ \left( |H_i g H_i \in H_i \setminus G/H_i \mid H_i g H_i \neq H_i g^{-1}H_i| \right) / 2.$$ 

End Do

We compute $L(G \times S_2)_i$ by using this algorithm with the function “GS2orbitnumber” in Program 5.4.

**Program 5.4.**
\begin{verbatim}
G := SL( 2, Integers mod 5 );
# Define the group SL(2,5).
# G := AlternatingGroup( 5 );
# Define the group A_5.
CCS := ConjugacyClassesSubgroups( G );
# the set of all conjugacy classes of element in G.
CCSR := List( CCS, H -> Representative(H) );
# the complete list [H_i] of representatives
# for conjugacy classes of subgroups of G.
N_G := List( CCSR, LC -> List(CCSR, RC ->
Length( DoubleCosets( G, LC, RC ) ) ) );
# the matrix of the number of double coset [H_i\G/H_j]
#(namely G-orbits in (G/H_i * G/H_j)).
N_GS2 := List( CCSR, ccsr -> GS2orbitnumber ( G, ccsr ) );
# the list of the number of (G*S_2)-orbits in
#(G/H_i * G/H_i).

# ccsr: the element of the complete set of representatives
#for conjugacy classes of subgroups of G.
# OUTPUT: the number of (G*S_2)-orbits in (G/H_i * G/H_i).
GS2orbitnumber := function( G, ccsr )
    local g, m, Temp, DC;
    Temp := [];
    m := 0;
    for g in G do
        DC := DoubleCoset( ccsr, g, ccsr );
        if not ( DC in Temp ) then
            Add( Temp, DC );
            if ( g^-1 in DC )
                then m := m + 1;
            else m := m + 1/2;
            fi;
        fi;
    od;
    return m;
end;

But use Program 5.4, we are able to only compute the number of orbit space. For more information (for example orbit-length), the next algorithm is useful.

Algorithm 5.5. Let \( G \) and \( \mathcal{R}_G = \{ H_i \mid i = 1, 2, \ldots, t \} \) be as above. \( \mathcal{M}(G) \) is the upper triangular matrix where \((i,j)\)-entry is the set of \( G \)-orbits contained in \( G/H_i \times G/H_j, \mathcal{L}(G \times \mathcal{G}_2) \) is the list of the set of \( G \times \mathcal{G}_2 \)-orbits contained in \( G/H_i \times G/H_i \).

Input: \( G \)
Output: \( \mathcal{M}(G), \mathcal{L}(G \times \mathcal{G}_2) \)

\( \mathcal{R}_G := \{ H_i \mid 1 \leq i \leq t \} \)

For \( 1 \leq i \leq t \) Do
    For \( i \leq j \leq t \) Do
        \( S := \emptyset \)
        For \( g \in G \) Do
            \( T := \) the \( G \)-orbit of \((eH_i, gH_j)\)
            If \( T \notin S \) Then \( S := S \cup \{ T \} \)
    od;
od;
end;
\end{verbatim}
\[ \mathcal{M}(G)_{ij} := S \]

For \( 1 \leq i \leq t \) Do
\[
\text{For } \{(x_j, y_j)\} \in \mathcal{M}(G)_{ii} \text{ Do}
\]
\[
T := \{(x_j, y_j)\} \cup \{(y_j, x_j)\}
\]
\[
\text{If } T \notin S \text{ Then } S := S \cup \{T\}
\]
End Do
\[ \mathcal{L}(G \times S_2)_i := S \]
End Do

In this Algorithm, we need check whether \( T \notin S \) for provided \( S = \{S_i\} \) and \( T \), namely check whether \( T \) coincides with an element of \( S \). Since \( T \) and \( S_i \) are \( G \)-orbits, \( T \cap S_i \neq \emptyset \Leftrightarrow T = S_i \). We use this property in programming the algorithm with GAP.

Program 5.6.

```plaintext
G := SL( 2, Integers mod 5 );
# Define the group SL(2,5).
# G := AlternatingGroup( 5 );
CCS := ConjugacyClassesSubgroups( G );
# the set of all conjugacy classes of element in G.
CCSR := List( CCS, H -> Representative( H ) );
# the complete list \([H_i]\) of representatives
# for conjugacy classes of subgroups of G.
CH := List( CCSR, ccsr -> RightCosets( G, ccsr ) );
# the list of G/H_i.
CHsize := Length( CH );
# the length of CH.
G_Orbit := List( [1..CHsize], k -> allorbit( G, CH, k ) );
# the upper triangular matrix of G-orbits in (G/H_i * G/H_j).
G_OS := List( G_Orbit, A -> List( A, B -> List( B, C ->
Length(C) ) ) );
# the upper triangular matrix of G-orbit-length in
#(G/H_i * G/H_j).
NG_OS := List( G_OS, gos -> Length( gos ) );
# the upper triangular matrix of the number of G-orbits in
#(G/H_i * G/H_j).
GS2_Orbit:=List([1..CHsize],i->S2Gorbit(G_Orbit[i][i]));
# the list of (G*S_2)-orbits in (G/H_i * G/H_i).
GS2_OS := List( GS2_Orbit, Gs -> List( Gs, gs ->
Length(gs) ) );
# the list of (G*S_2)-orbit-length in (G/H_i * G/H_i).
NGS2_OS :=List(GS2_OS,s->Length(s));
# the list of the number of (G*S_2)-orbits in (G/H_i * G/H_i).
```

# ot: an orbit,
# OT: a list of orbits.
# OUTPUT: if ot in OT, then return true.
orbitin := function( ot, OT )
local r;
for r in OT do
```
```
if ot[1] in r then
    return true;
fi;
odd;
return false;
end;

# cosetl: a coset of left part in (G/H_i * G/H_j),
# cosetr: a coset of right part in (G/H_i * G/H_j).
# OUTPUT: G-orbit G(aH, bK).
Gorbit := function( G, cosetl, cosetr )
local g, Temp, Result;

Result := [];
for g in G do
    Temp := [ cosetl*g, cosetr*g ];
    if not (Temp in Result) then
        Add( Result, Temp );
    fi;
od;
return Result;
end;

# GOH: G/H where H is a subgroup of G,
# GOK: G/K where K is a subgroup of G.
# OUTPUT: the list of G-orbit in (G/H * G/K).
Gorbitspace := function( G, GOH, GOK )
local gK, Result, Temp;

Result := [];
for gK in GOK do
    Temp := Gorbit( G, GOH[1], gK );
    if not orbitin( Temp, Result ) then
        Add( Result, Temp );
    fi;
od;
return Result;
end;

# i: i-th row.
# OUTPUT: a row of the upper triangular matrix of G-orbit
# space G\(G/H_i \ast G/H_j)\).
allorbit := function( G, CH, i )
local j, Result;

Result := [];
for j in [i..Length( CH )] do
    Result[j] := Gorbitspace( G, CH[i], CH[j] );
od;
return Result;
end;

# GODC: G-orbit in (G/H_i \ast G/H_i).
# OUTPUT: (G*S_2)-orbits in (G/H_i \ast G/H_i).
S2Gorbit:=function(GODC)
local i, s, Result, Temp;

s:=Length(GODC);
Result:=[ ];
for i in [1..s] do
    Temp := List([1..s], j -> [GODC[i][j][2], GODC[i][j][1]]);
    if not orbitin(Temp, Result) then
        Add(Result, GODC[i]);
    fi;
od;
return Result;
end;

References


