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Computation of the Grothendieck-Witt ring of nonsingular Hermitian forms with positioning map

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The generalized Grothendieck-Witt ring of nonsingular Hermitian forms with positioning map is the cartesian product of the ordinary Grothendieck-Witt ring of nonsingular Hermitian forms and a map ring. We study the generalized Grothendieck-Witt ring by computing the map ring.

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1 Introduction

Throughout this paper, let G be a finite group, R a commutative ring with 1, and Θ a finite G-set. The ordinary Grothendieck-Witt ring $\mathrm{GW}_0(R,G)$ in A. Dress [10] was generalized to $\mathrm{GW}_0(R,G,\Theta)$ in [5, p. 7], [13, p. 506], [16, p. 2356], [7, pp. 59–61], and the classical one is obtained as the case $\Theta = \emptyset$, namely $\mathrm{GW}_0(R,G) = \mathrm{GW}_0(R,G,\emptyset)$. Let $\mathfrak{S}_2$ denote the symmetric group of degree 2 and τ the generator of $\mathfrak{S}_2$. The cartesian product $\Theta \times \Theta$ is afforded the $G \times \mathfrak{S}_2$-action such that $g(t,t') = (gt, gt')$ and $\tau(t,t') = (t',t)$ for $g \in G$ and t, $t' \in \Theta$. Let $\mathrm{Map}_G(\Theta \times \Theta, R)$ (resp. $\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R)$) denote the set of all G-invariant (resp. $G \times \mathfrak{S}_2$-invariant) maps from $\Theta \times \Theta$ to R. This set $\mathrm{Map}_G(\Theta \times \Theta, R)$ (resp. $\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R)$) has a canonical ring structure and isomorphic to $\mathrm{Map}(G\backslash(\Theta \times \Theta), R)$ (resp. $\mathrm{Map}((G \times \mathfrak{S}_2)\backslash(\Theta \times \Theta), R)$), where $G\backslash(\Theta \times \Theta)$ (resp. $(G \times \mathfrak{S}_2)\backslash(\Theta \times \Theta)$) is the G-orbit space (resp. $G \times \mathfrak{S}_2$-orbit space) of $\Theta \times \Theta$. We define

$$M(G,\Theta,R) = \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R).$$

If R is a principal ideal domain then by [7] $\mathrm{GW}_0(R,G,\Theta)$ is isomorphic to the cartesian product of $\mathrm{GW}_0(R,G)$ and $M(G,\Theta,R)$ as rings. Let $\mathrm{RO}(G)$ denote the real representation ring of G.

Theorem 1.1. *The abelian group* $\mathbb{Z}[\frac{1}{2}] \otimes \mathrm{GW}_0(\mathbb{Z},G,\Theta)$ *is isomorphic to*

$$\mathbb{Z}[\frac{1}{2}] \otimes \mathrm{RO}(G) \oplus M(G,\Theta,\mathbb{Z}[\frac{1}{2}]).$$

The $\mathbb{Z}$*-rank of* $\mathrm{GW}_0(\mathbb{Z},G,\Theta)$ *is the sum of the number of isomorphism classes of finite dimensional irreducible real* G*-modules and the* $\mathbb{Z}$*-rank of* $M(G,\Theta,\mathbb{Z})$*. Moreover, the torsion subgroup of* $\mathrm{GW}_0(\mathbb{Z},G,\Theta)$ *is annihilated by* 4.

Theorem 1.2. *Let* Θ *be a finite* G*-set isomorphic to* $\coprod_{i=1}^{t}\left(\coprod_{j=1}^{a_i} G/H_i\right)$ $(a_i \in \mathbb{N})$*. Then* $M(G,\Theta,R)$ *is an* R*-free module of rank*

$$\sum_{i=1}^{t}\Big(a_i\big|(G \times \mathfrak{S}_2)\backslash(G/H_i \times G/H_i)\big| + \frac{a_i^2 - a_i}{2}\big|G\backslash(G/H_i \times G/H_i)\big|\Big) + \sum_{1\le i<j\le t} a_i a_j \big|G\backslash(G/H_i \times G/H_j)\big|,$$

where $\big|(G\times\mathfrak{S}_2)\backslash(G/H_i \times G/H_i)\big|$ *stands for the number of* $G\times\mathfrak{S}_2$*-orbits in* $G/H_i \times G/H_i$*, and* $\big|G\backslash(G/H_i \times G/H_j)\big|$ *stands for the number of* G*-orbits in* $G/H_i \times G/H_j$.

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As applications of these theorems, we compute the $\mathbb{Z}$-rank of $\mathrm{GW}_0(\mathbb{Z}, G, \Theta)$ in the cases $G = A_5$, the alternating group of degree 5, and $\mathrm{SL}(2, \mathbb{Z}_5)$, the special linear group of degree 2 with coefficients in $\mathbb{Z}_5$. The results are described in Sections 2 and 3.

2 The rank of for $\mathrm{GW}_0(\mathbb{Z}, G, \Theta)$ for $G = A_5$

A complete set $\mathcal{R}_{A_5}$ of representatives for conjugacy classes of subgroups of $G = A_5$ is

$$\mathcal{R}_{A_5} = \{\{e\},\ C_2,\ C_3,\ D_4,\ C_5,\ D_6,\ D_{10},\ A_4,\ A_5\},$$

where $\{e\}$ is the unit group, C_n $(n = 2, 3, 5)$ are the cyclic groups of order n, generated by the elements σ_n:

$$\sigma_2 = (1,2)(3,4), \quad \sigma_3 = (1,2,3), \quad \sigma_5 = (1,2,3,4,5),$$

D_{2n} are the dihedral groups of order $2n$ obtained by $D_{2n} = N_G(C_n)$, and A_4 is the alternating group on four letters $\{1,2,3,4\}$.

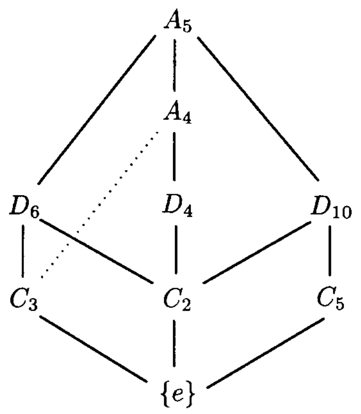

Figure 1: Conjugacy Class Subgroups of A_5

Lemma 2.1. *In the case $G = A_5$, $\mathrm{RO}(G)$ is a free $\mathbb{Z}$-module of rank 5.*

Proof. This follows from [8, Theorem 38.1] and Frobenius-Shur's Theorem, since $A_5 = \mathrm{SL}(2, \mathbb{Z}_5)/C_2$. □

Let Θ be a finite G-set. A $G \times \mathfrak{S}_2$-orbit X in $\Theta \times \Theta$ is called $\mathfrak{S}_2$*-effective* if X contains precisely two G-orbits, and $\mathfrak{S}_2$*-indecomposable* if X contains only one G-orbit.

The Program 5.6 for $G = A_5$ given in Section 5 provides the next result which was theoretically computed by M. Kubo [12].

Lemma 2.2. *Let $G = A_5$ and $\mathcal{R}_{A_5} = \{H_i \mid i = 1, 2, \cdots, 9\}$ the complete set above of representatives for conjugacy classes of subgroups of A_5. Then the number of G-orbits in $G/H_i \times G/H_j$ is given as in Table 1 and the number of $G \times \mathfrak{S}_2$-orbits in $G/H_i \times G/H_i$ is given as in Table 2.*

The notation in the tables reads that, for example in the case

$$2_{20\times 1} + 4_{60\times 1} + 1_{60\times 2},$$

there exist two $\mathfrak{S}_2$-indecomposable $G \times \mathfrak{S}_2$-orbits of orbit-length 20, four $\mathfrak{S}_2$-indecomposable $G \times \mathfrak{S}_2$-orbits of orbit-length 60 and one $\mathfrak{S}_2$-decomposable $G \times \mathfrak{S}_2$-orbit of orbit-length 120.

The next theorem immediately follows from Theorem 2.2.

	$\{e\}$	C_2	C_3	D_4	C_5	D_6	D_{10}	A_4	A_5
$\{e\}$	60_{60}	30_{60}	20_{60}	15_{60}	12_{60}	10_{60}	6_{60}	5_{60}	1_{60}
C_2		$2_{30}+14_{60}$	10_{60}	$3_{30}+6_{60}$	6_{60}	$2_{30}+4_{60}$	$2_{30}+2_{60}$	$1_{30}+2_{60}$	1_{30}
C_3			$2_{20}+6_{60}$	6_{60}	4_{60}	$1_{20}+3_{60}$	2_{60}	$2_{20}+1_{60}$	1_{20}
D_4				$3_{15}+3_{60}$	3_{60}	$3_{30}+1_{60}$	3_{30}	$1_{15}+1_{60}$	1_{15}
C_5					$2_{12}+2_{60}$	2_{60}	$1_{12}+1_{60}$	1_{60}	1_{12}
D_6						$1_{10}+1_{30}+1_{60}$	2_{30}	$1_{20}+1_{30}$	1_{10}
D_{10}							$1_{6}+1_{30}$	1_{30}	1_{6}
A_4								$1_{5}+1_{20}$	1_{5}
A_5									1_{1}

Table 1: The number of G-orbits in $G/H_i \times G/H_j$ where $G = A_5$

H_i	orbits
$\{e\}$	$16_{60\times1}+22_{60\times2}$
C_2	$2_{30\times1}+6_{60\times1}+4_{60\times2}$
C_3	$2_{20\times1}+4_{60\times1}+1_{60\times2}$
D_4	$1_{15\times1}+1_{15\times2}+3_{60\times1}$
C_5	$2_{12\times1}+2_{60\times1}$
D_6	$1_{10\times1}+1_{30\times1}+1_{60\times1}$
D_{10}	$1_{6\times1}+1_{30\times1}$
A_4	$1_{5\times1}+1_{20\times1}$
A_5	$1_{1\times1}$

Table 2: The number of $G \times \mathfrak{S}_2$-orbits in $G/H_i \times G/H_i$ where $G = A_5$

Theorem 2.3. *Let G and $\mathcal{R}_{A_5} = \{H_i \mid i = 1, 2, \cdots, 9\}$ be as above. Then the numbers $|(G\times\mathfrak{S}_2)\backslash(G/H_i \times G/H_i)|$ and $|G\backslash(G/H_i \times G/H_j)|$ appearing in Theorem 1.2 are given as in Tables 3 and 4, respectively.*

$\{e\}$	C_2	C_3	D_4	C_5	D_6	D_{10}	A_4	A_5
38	12	7	5	4	3	2	2	1

Table 3: $|(G\times\mathfrak{S}_2)\backslash(G/H_i \times G/H_i)|$ where $G = A_5$

	$\{e\}$	C_2	C_3	D_4	C_5	D_6	D_{10}	A_4	A_5
$\{e\}$	60	30	20	15	12	10	6	5	1
C_2		16	10	9	6	6	4	3	1
C_3			8	6	4	4	2	3	1
D_4				6	3	4	3	2	1
C_5					4	2	2	1	1
D_6						3	2	2	1
D_{10}							2	1	1
A_4								2	1
A_5									1

Table 4: $|G\backslash(G/H_i \times G/H_j)|$ where $G = A_5$

3 The rank of for $\mathrm{GW}_0(\mathbb{Z}, G, \Theta)$ for $G = \mathrm{SL}(2, \mathbb{Z}_5)$

We recall basic properties of $\mathrm{SL}(2,\mathbb{Z}_5)$, the special linear group of degree 2 on $\mathbb{Z}_5$. A complete set $\mathcal{R}_{\mathrm{SL}(2,\mathbb{Z}_5)}$ of representatives for conjugacy classes of subgroups of $\mathrm{SL}(2,\mathbb{Z}_5)$ is given by

$$\mathcal{R}_{\mathrm{SL}(2,\mathbb{Z}_5)} = \{\{e\},\ C_2,\ C_3,\ C_4,\ C_5,\ C_6,\ Q_8,\ C_{10},\ Q_{12},\ Q_{20},\ \pi^{-1}(A_4),\ \mathrm{SL}(2,\mathbb{Z}_5)\},$$

where $\{e\}$ is unit group, C_n (n = 2, 3, 4, 5, 6, 10) are the cyclic groups of order n, Q_{4m} (n = 2, 3, 5) are the quaternion groups of order $4m$, namely

$$Q_{4m} = \langle a, b \mid b^m = a^2, b^{2m} = e, a^{-1}ba = b^{-1}\rangle,$$

and

$$\pi : \mathrm{SL}(2,\mathbb{Z}_5) \to \mathrm{SL}(2,\mathbb{Z}_5)/C_2\ (= A_5);\ \ a \to aC_2.$$

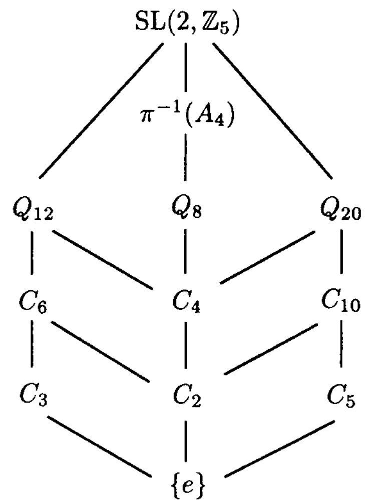

Figure 2: Conjugacy Class Subgroups of $\mathrm{SL}(2,\mathbb{Z}_5)$

Lemma 3.1. *In the case $G = \mathrm{SL}(2,\mathbb{Z}_5)$, $\mathrm{RO}(G)$ is a free $\mathbb{Z}$-module of rank 9.*

Proof. This follows from [8, Theorem 38.1] and Frobenius-Shur's Theorem. □

Lemma 3.2. *Let $G = \mathrm{SL}(2,\mathbb{Z}_5)$ and $\mathcal{R}_{\mathrm{SL}(2,\mathbb{Z}_5)} = \{H_i \mid i = 1, 2, \cdots, 12\}$ the complete set above of representatives for conjugacy classes of subgroups of $\mathrm{SL}(2,\mathbb{Z}_5)$. Then the number of G-orbits in $G/H_i \times G/H_j$ is given as in Table 5 and the number of $G \times \mathfrak{S}_2$-orbits in $G/H_i \times G/H_i$ is given as in Table 6.*

The next theorem immediately follows from Theorem 3.2.

Theorem 3.3. *Let G and $\mathcal{R}_{\mathrm{SL}(2,\mathbb{Z}_5)} = \{H_i \mid i = 1, 2, \cdots, 12\}$ be as above. Then the numbers $|(G \times \mathfrak{S}_2)\backslash(G/H_i \times G/H_i)|$ and $|G\backslash(G/H_i \times G/H_j)|$ appearing in Theorem 1.2 are given as in Tables 7 and 8, respectively.*

4 Proofs of Theorems 1.1 and 1.2

The ordinary Grothendieck-Witt ring $\mathrm{GW}_0(\mathbb{Z}, G)$ is defined in [9, p. 742], and the Grothendieck-Witt ring $\mathrm{GW}_0(R, G, \Theta)$ is defined in [5], [13], [16, §4] and [7, pp. 59–61].

Proof of Theorem 1.1. If R is a principal ideal domain then by Theorem 1 of [7]

$$\mathrm{GW}_0(R, G, \Theta) \cong \mathrm{GW}_0(R, G) \oplus M(G, \Theta, R).$$

By definition, $M(G,\Theta,R) = \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R)$ is a free R-module. Thus Theorem 1.1 follows from [9, Theorem 3]. □

	$\{e\}$	C_2	C_3	C_4	C_5	C_6	Q_8	C_{10}	Q_{12}	Q_{20}	$\pi^{-1}(A_4)$	$\mathrm{SL}(2,\mathbb{Z}_5)$
$\{e\}$	120_{120}	60_{120}	40_{120}	30_{120}	24_{120}	20_{120}	15_{120}	12_{120}	10_{120}	6_{120}	5_{120}	1_{120}
C_2		60_{60}	20_{120}	30_{60}	12_{120}	20_{60}	15_{60}	12_{60}	10_{60}	6_{60}	5_{60}	1_{60}
C_3			$4_{40}+12_{120}$	10_{120}	8_{120}	$2_{40}+6_{120}$	5_{120}	4_{120}	$1_{40}+3_{120}$	2_{120}	$2_{40}+1_{120}$	1_{40}
C_4				$2_{30}+14_{60}$	6_{120}	10_{60}	$3_{30}+6_{60}$	6_{60}	$2_{30}+4_{60}$	$2_{30}+2_{60}$	$1_{30}+2_{60}$	1_{30}
C_5					$4_{24}+4_{120}$	4_{120}	3_{120}	$2_{24}+2_{120}$	2_{120}	$1_{24}+1_{120}$	1_{120}	1_{24}
C_6						$2_{20}+6_{60}$	5_{60}	4_{60}	$1_{20}+3_{60}$	2_{60}	$2_{20}+1_{60}$	1_{20}
Q_8							$3_{15}+3_{60}$	3_{60}	$3_{30}+1_{60}$	3_{30}	$1_{15}+1_{60}$	1_{15}
C_{10}								$2_{12}+2_{60}$	2_{60}	$1_{12}+1_{60}$	1_{60}	1_{12}
Q_{12}									$1_{10}+1_{30}+1_{60}$	2_{30}	$1_{20}+1_{30}$	1_{10}
Q_{20}										1_6+1_{30}	1_{30}	1_6
$\pi^{-1}(A_4)$											1_5+1_{20}	1_5
$\mathrm{SL}(2,\mathbb{Z}_5)$												1_1

Table 5: The number of G-orbits in $G/H_i \times G/H_j$ where $G=\mathrm{SL}(2,\mathbb{Z}_5)$

H_i	orbits
$\{e\}$	$2_{120\times1}+59_{120\times2}$
C_2	$16_{60\times1}+22_{60\times2}$
C_3	$2_{40\times1}+1_{40\times2}+6_{120\times2}$
C_4	$2_{30\times1}+6_{60\times1}+4_{60\times2}$
C_5	$2_{24\times1}+1_{24\times2}+2_{120\times2}$
C_6	$2_{20\times1}+4_{60\times1}+1_{60\times2}$
Q_8	$1_{15\times1}+1_{15\times2}+3_{60\times1}$
C_{10}	$2_{12\times1}+2_{60\times1}$
Q_{12}	$1_{10\times1}+1_{30\times1}+1_{60\times1}$
Q_{20}	$1_{6\times1}+1_{30\times1}$
$\pi^{-1}(A_4)$	$1_{5\times1}+1_{20\times1}$
$\mathrm{SL}(2,\mathbb{Z}_5)$	$1_{1\times1}$

Table 6: The number of $G\times\mathfrak{S}_2$-orbits in $G/H_i \times G/H_i$ where $G=\mathrm{SL}(2,\mathbb{Z}_5)$

$\{e\}$	C_2	C_3	C_4	C_5	C_6	Q_8	C_{10}	Q_{12}	Q_{20}	$\pi^{-1}(A_4)$	$\mathrm{SL}(2,\mathbb{Z}_5)$
61	38	9	12	5	7	5	4	3	2	2	1

Table 7: $\left|(G\times\mathfrak{S}_2)\backslash(G/H_i\times G/H_i)\right|$ where $G=\mathrm{SL}(2,\mathbb{Z}_5)$

	$\{e\}$	C_2	C_3	C_4	C_5	C_6	Q_8	C_{10}	Q_{12}	Q_{20}	$\pi^{-1}(A_4)$	$\mathrm{SL}(2,\mathbb{Z}_5)$
$\{e\}$	120	60	40	30	24	20	15	12	10	6	5	1
C_2		60	20	30	12	20	15	12	10	6	5	1
C_3			16	10	8	8	5	4	4	2	3	1
C_4				16	6	10	9	6	6	4	3	1
C_5					8	4	3	4	2	2	1	1
C_6						8	5	4	4	2	3	1
Q_8							6	3	4	3	2	1
C_{10}								4	2	2	1	1
Q_{12}									3	2	2	1
Q_{20}										2	1	1
$\pi^{-1}(A_4)$											2	1
$\mathrm{SL}(2,\mathbb{Z}_5)$												1

Table 8: $|G\backslash(G/H_i \times G/H_j)|$ where $G = \mathrm{SL}(2,\mathbb{Z}_5)$

Proof of Theorem 1.2. Since

$$\mathrm{Map}_{G\times\mathfrak{S}_2}((\Theta_1 \amalg \Theta_2) \times (\Theta_1 \amalg \Theta_2), R) \cong \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta_1 \times \Theta_1, R) \oplus \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta_2 \times \Theta_2, R) \oplus \mathrm{Map}_G(\Theta_1 \times \Theta_2, R),$$

we obtain

$$\mathrm{Map}_{G\times\mathfrak{S}_2}((\coprod_{i=1}^{t} \Theta_i) \times (\coprod_{i=1}^{t} \Theta_i), R) \cong \bigoplus_{i=1}^{t} \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta_i \times \Theta_i, R) \oplus \bigoplus_{1\le i<j\le t} \mathrm{Map}_G(\Theta_i \times \Theta_j, R).$$

The next Lemma follows from this.

Lemma 4.1. *If* $\Theta \cong \coprod_{i=1}^{t}(\coprod_{j=1}^{a_i} G/H_i)$ $(a_i \in \mathbb{N})$ *then*

$$M(G,\Theta,R) \cong \bigoplus_{i=1}^{t}\left\{\bigoplus_{k=1}^{a_i} M(G,G/H_i,R) \oplus \bigoplus_{k=1}^{(a_i^2-a_i)/2} \mathrm{Map}_G(G/H_i \times G/H_i, R)\right\} \oplus \bigoplus_{1\le i<j\le t}\left\{\bigoplus_{k=1}^{a_i\times a_j} \mathrm{Map}_G(G/H_i \times G/H_j, R)\right\}.$$

Let $\mathcal{R}_G = \{H_i \mid 1 \le i \le t\}$ be a complete set of representatives for conjugacy classes of subgroups of G. By the Lemma above, $\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta)$ is determined by the numbers of elements in $G \setminus (G/H_i \times G/H_j)$ and $(G \times \mathfrak{S}_2) \setminus (G/H_i \times G/H_i)$, where H_i and H_j run over $\mathcal{R}_G$. □

5 Appendix: Algorithms and Programmings

In the Appendix, we give the algorithms and programmings for Lemma 2.2 and Lemma 3.2.

Lemma 5.1. *Let H and K be subgroups of G. Then the following hold.*

(1) $G/H \times G/K$ has the G-orbit decomposition

$$\coprod_{HgK\in H\backslash G/K} G\cdot(eH, gK).$$

(2) Each G-orbit $G\cdot(H, gH)$ is transposed to the G-orbit $G\cdot(H, g^{-1}H)$ by the generator τ of $\mathfrak{S}_2$.

(3) A G-orbit $G\cdot(H,gH)$ is itself a $G\times\mathfrak{S}_2$-orbit if and only if $HgH=Hg^{-1}H$.

This lemma is well known but we give the proof for the reader's convenience.

Proof. (1) Each G-orbit in $G/H\times G/K$ has a representative of the form (eH,gK). The points (eH,aK) and (eH,bK), where a and $b\in G$, lie in a same G-orbit if and only if $HaK=HbK$. Thus we get

$$G/H\times G/K=\coprod_{HgK\in H\backslash G/K} G\cdot(eH,gK).$$

(2) The claim follows from $\tau(eH,gK)=(gH,eK)$ and $G\cdot(eH,gH)=G\cdot(eH,g^{-1}H)$.

(3) Since $(G\times\mathfrak{S}_2)\cdot(eH,gH)=G\cdot(eH,gH)\cup G\cdot(eH,g^{-1}H)$,

$$(G\times\mathfrak{S}_2)\cdot(eH,gH)=G\cdot(eH,gH)$$

holds if and only if $G\cdot(eH,gH)=G\cdot(eH,g^{-1}H)$, in other words $HgH=Hg^{-1}H$. □

Proposition 5.2. *One has the formulae:*

(1) $|G\backslash(G/H\times G/K)|=|H\backslash G/K|$,

(2) $$|(G\times\mathfrak{S}_2)\backslash(G/H\times G/H)|=|\{HgH\in H\backslash G/H\mid HgH=Hg^{-1}H\}| + \frac{|\{HgH\in H\backslash G/H\mid HgH\neq Hg^{-1}H\}|}{2}.$$

Proof. This immediately follows from Lemma 5.1. □

We use GAP (Groups, Algorithms, and Programming) to compute the number of orbits $|G\backslash(G/H_i\times G/H_j)|$ and $|(G\times\mathfrak{S}_2)\backslash(G/H_i\times G/H_i)|$ for $G=A_5$ and $\mathrm{SL}(2,\mathbb{Z}_5)$ together with Proposition 5.2 (see Algorithm 5.3 and Program 5.4).

Algorithm 5.3. Let G be a finite group, $\mathcal{R}_G=\{H_i\mid i=1,2,\cdots,t\}$ a complete set of representatives for conjugacy classes of subgroups of G. $\mathrm{M}(G)$ will be the matrix $(|H_i\backslash G/H_j|)$ (namely $(|G\backslash(G/H_i\times G/H_j)|)$), and $\mathrm{L}(G\times\mathfrak{S}_2)$ will be the list of $|(G\times\mathfrak{S}_2)\backslash(G/H_i\times G/H_i)|$.

> **Input:** G
> **Output:** $\mathrm{M}(G)$, $\mathrm{L}(G\times\mathfrak{S}_2)$
>
> $\mathcal{R}_G:=\{H_i\mid 1\le i\le t\}$
>
> **For** $1\le i\le t$ **Do**
>   **For** $1\le j\le t$ **Do**
> $\mathrm{M}(G)_{ij}:=|H_i\backslash G/H_j|$
>   **End Do**
> **End Do**
>
> **For** $1\le i\le t$ **Do**
> $\mathrm{L}(G\times\mathfrak{S}_2)_i:=|\{H_igH_i\in H_i\backslash G/H_i\mid H_igH_i=H_ig^{-1}H_i\}| + \frac{|\{H_igH_i\in H_i\backslash G/H_i\mid H_igH_i\neq H_ig^{-1}H_i\}|}{2}$
> **End Do**

We compute $\mathrm{L}(G\times\mathfrak{S}_2)_i$ by using this algorithm with the function "GS2orbitnumber" in Program 5.4.

Program 5.4.

```
G := SL( 2, Integers mod 5 );
# Define the group SL(2,5).
# G := AlternatingGroup( 5 );
# Define the group A_5.
CCS := ConjugacyClassesSubgroups( G );
# the set of all conjugacy classes of element in G.
CCSR := List( CCS, H -> Representative(H) );
# the complete list [H_i] of representatives
#for conjugacy classes of subgroups of G.
N_G := List( CCSR, LC -> List(CCSR, RC ->
Length( DoubleCosets( G, LC, RC ) ) ) );
# the matrix of the number of double coset [H_i\G/H_j]
#(namely G-orbits in (G/H_i * G/H_j)).
N_GS2 := List( CCSR, ccsr -> GS2orbitnumber ( G, ccsr ) );
# the list of the number of (G*S_2)-orbits in
#(G/H_i * G/H_i).


# ccsr: the element of the complete set of representatives
#for conjugacy classes of subgroups of G.
# OUTPUT: the number of (G*S_2)-orbits in (G/H_i * G/H_i).
GS2orbitnumber := function( G, ccsr )
     local g, m, Temp, DC;

     Temp := [];
     m := 0;
     for g in G do
          DC := DoubleCoset( ccsr, g, ccsr );
          if not ( DC in Temp ) then
               Add( Temp, DC );
               if ( g^-1 in DC )
               then m := m + 1;
               else m := m + 1/2;
               fi;
          fi;
     od;
     return m;
end;
```

But use Program 5.4, we are able to only compute the number of orbit space. For more information (for example orbit-length), the next algorithm is useful.

Algorithm 5.5. Let G and $\mathcal{R}_G = \{H_i \mid i = 1, 2, \cdots, t\}$ be as above. $\mathcal{M}(G)$ is the upper triangular matrix where (i, j)-entry is the set of G-orbits contained in $G/H_i \times G/H_j$, $\mathcal{L}(G \times \mathfrak{S}_2)$ is the list of the set of $G \times \mathfrak{S}_2$-orbits contained in $G/H_i \times G/H_i$.

Input: G
Output: $\mathcal{M}(G)$, $\mathcal{L}(G \times \mathfrak{S}_2)$

$\mathcal{R}_G := \{H_i \mid 1 \le i \le t\}$

For $1 \le i \le t$ **Do**
  For $i \le j \le t$ **Do**
    $S := \emptyset$
    For $g \in G$ **Do**
      $T :=$ the G-orbit of (eH_i, gH_j)
      If $T \notin S$ **Then** $S := S \cup \{T\}$

End Do
$\mathcal{M}(G)_{ij} := S$
End Do
End Do

For $1 \leq i \leq t$ **Do**
$S := \emptyset$
For $\{(x_j, y_j)\} \in \mathcal{M}(G)_{ii}$ **Do**
$T := \{(x_j, y_j)\} \cup \{(y_j, x_j)\}$
If $T \notin S$ **Then** $S := S \cup \{T\}$
End Do
$\mathcal{L}(G \times \mathfrak{S}_2)_i := S$
End Do

In this Algorithm, we need check whether $T \notin S$ for provided $S = \{S_i\}$ and T, namely check whether T coincides with an element of S. Since T and S_i are G-orbits, $T \cap S_i \neq \emptyset \Leftrightarrow T = S_i$. We use this property in programming the algorithm with GAP.

Program 5.6.

```
G := SL( 2, Integers mod 5 );
# Define the group SL(2,5).
# G := AlternatingGroup( 5 );
CCS := ConjugacyClassesSubgroups( G );
# the set of all conjugacy classes of element in G.
CCSR := List( CCS, H -> Representative( H ) );
# the complete list [H_i] of representatives
#for conjugacy classes of subgroups of G.
CH := List( CCSR, ccsr -> RightCosets( G, ccsr ) );
# the list of G/H_i.
CHsize := Length( CH );
# the length of CH.
G_Orbit := List( [1..CHsize], k -> allorbit( G, CH, k ) );
# the upper triangular matrix of G-orbits in (G/H_i * G/H_j).
G_OS := List( G_Orbit, A -> List( A, B -> List( B, C ->
Length(C) ) ) );
# the upper triangular matrix of G-orbit-length in
#(G/H_i * G/H_j).
NG_OS := List( G_OS, Gos -> List( Gos, gos -> Length(gos) ) );
# the upper triangular matrix of the number of G-orbits in
# (G/H_i * G/H_j).
GS2_Orbit:=List([1..CHsize],i->S2Gorbit(G_Orbit[i][i]));
# the list of (G*S_2)-orbits in (G/H_i * G/H_i).
GS2_OS := List( GS2_Orbit, Gs -> List( Gs, gs ->
Length(gs) ) );
# the list of (G*S_2)-orbit-length in (G/H_i * G/H_i).
NGS2_OS :=List(GS2_OS,s->Length(s));
# the list of the number of (G*S_2)-orbits in (G/H_i * G/H_i).


# ot: an orbit,
# OT: a list of orbits.
# OUTPUT: if ot in OT, then return true.
orbitin := function( ot, OT )
     local r;

     for r in OT do
```

```
            if ot[1] in r then
                return true;
            fi;
        od;
        return false;
end;

# cosetl: a coset of left part in (G/H_i * G/H_j),
# cosetr: a coset of right part in (G/H_i * G/H_j).
# OUTPUT: G-orbit G(aH, bK).
Gorbit := function( G, cosetl, cosetr )
        local g, Temp, Result;

        Result := [];
        for g in G do
            Temp := [ cosetl*g, cosetr*g ];
            if not (Temp in Result) then
                Add( Result, Temp );
            fi;
        od;
        return Result;
end;

# GOH: G/H where H is a subgroup of G,
# GOK: G/K where K is a subgroup of G.
# OUTPUT: the list of G-orbit in (G/H * G/K).
Gorbitspace := function( G, GOH, GOK )
        local gK, Result, Temp;

        Result := [];
        for gK in GOK do
            Temp := Gorbit( G, GOH[1], gK );
            if not orbitin( Temp, Result ) then
                Add( Result, Temp );
            fi;
        od;
        return Result;
end;

# i: i-th row.
# OUTPUT: a row of the upper triangular matrix of G-orbit
#space G\(G/H_i * G/H_j).
allorbit := function( G, CH, i )
        local j, Result;

        Result := [];
        for j in [i..Length( CH )] do
            Result[j] := Gorbitspace( G, CH[i], CH[j] );
        od;
        return Result;
end;

# GODC: G-orbit in (G/H_i * G/H_i).
# OUTPUT: (G*S_2)-orbits in (G/H_i * G/H_i).
S2Gorbit:=function(GODC)
```

```
    local i, s, Result, Temp;

    s:=Length( GODC );
    Result:=[];
    for i in [1..s] do
        Temp := List( [1..s], j -> [ GODC[i][j][2],
GODC[i][j][1] ] );
        if not orbitin( Temp, Result ) then
            Add( Result, GODC[i] );
        fi;
    od;
    return Result;
end;
```

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