Two dimensional homogenized models of steel fiber reinforced concrete

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The homogenization method is used to model steel fiber reinforced concrete SFRC by converting the random distribution of fibers to a periodic one. The periodic distribution is chosen to hold similar properties of the composite material in both perpendicular directions to represent an average approximation for the random distribution. The material is modeled as a composite with brittle matrix and elastic fibers. Two patterns of the unit cell are examined to establish the homogenized stiffness matrix in elastic and plastic stages. A rigid plastic bonding is considered between matrix and fibers. The smeared crack model is used to represent the nonlinearity of concrete. The validity of the homogenized model is examined by comparing the numerical results with the experimental results. The results show good agreement with the experimental work when a suitable pattern of the unit cell is used.

Key words: fiber reinforced concrete, composite materials, homogenization, periodicity, finite element.

I INTRODUCTION

Steel Fiber Reinforced Concrete (SFRC) is concrete made primarily of cement, aggregates and discrete steel reinforcing fibers. The role of the randomly oriented discrete discontinuous fibers is to bridge across the cracks that develop in concrete either as it is loaded or as it is subjected to environmental changes. SFRC is today firmly established as a construction material. One of its greatest benefits is to improve the long-term serviceability of the structure. Serviceability is the ability of the specific structure to maintain its strength and integrity and to provide its designed function over its intended service life. Although there are many experimental works which examined the properties of SFRC, the numerical model for such material is still lacking.

This paper introduces a numerical solution to predict the flexural strength of SFRC using the so-called Homogenization method. Since the early studies of composite materials by microscopic observation in the 1950s, recent rapid progress in computer technologies has enabled us to analyze the microscopic mechanical behaviors numerically. As a theoretical and numerical technique to solve micro-macro coupling problems, the homogenization method is becoming more and more important for the understanding of micromechanics of composite materials. This method is an applied mathematical theory first developed in the early 1980s. Guedes and Kikuchi [1990] have published a pioneering work in the engineering field of metal matrix composite material. Hassani [1998], Takano [1999] and Terada [2000] developed the homogenization technique to solve many applications of composite materials. From these works, it is recognized that the homogenization method can predict the effective elastic properties with enough accuracy for arbitrary complex microstructures.

SFRC is also a composite material which can be solved by the homogenization method. The short fibers are usually scattered in a random way difficult to model. One way to overcome this difficulty is the introduction of the homogenization method which requires periodic distribution of fibers. Therefore the random distribution needs to be transformed to an equivalent periodic transformation which can be analyzed by the homogenization method.

This paper introduces 2 patterns of periodic distribution for SFRC to represent the unit cell of the structure. The boundary conditions for the unit cell and the homogenized stiffness matrix have been derived. The behavior of the composite material under flexural loading is studied before and after cracking. The paper
also studies the effect of fiber volume fraction on the toughness of concrete. The calculated results are compared with the experimental work of Yin et al [2003].

2. HOMOGENIZATION METHOD

The short steel fibers are usually distributed randomly in the concrete structures as it is shown in Fig.1-a. This random distribution makes the material unhomogenized. To solve these structures with the homogenization methods, we need to convert the random distribution to an equivalent periodic distribution.

If the fiber distribution is random we can expect physical behaviors along two perpendicular axes in a 2-dimensional area are same. So a symmetric pattern for periodicity is needed. Fig.1-b shows one way of the periodic distribution of fibers. The distance between fibers is calculated according to the required volume content of fibers. Two possible patterns P1 and P2 can be chosen to represent the unit cell as shown in Fig.1-b and Fig.2.

The equilibrium equation for an elastic problem is described as follows; refer to references (Hassani et al [1998], Takano et al [1999]) for more details. \( D_{ijkl} \) is an elastic tensor and \( t_i \) is the traction applied on the boundary \( \Gamma \).

\[
\int D_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_j} \, d\Omega = \int t_i u_i d\Gamma
\]  
(1)

The governing equation for homogenization is

\[
\int D_{ijkl} \frac{\partial x_{kl}}{\partial y_p} \frac{\partial u_i}{\partial y_j} \, dY = \int D_{ijkl} \frac{\partial u_i}{\partial y_j} \, dY
\]  
(2)

where \( Y \) is the volume of the microscopic unit, \( u_i \) is the microscopic displacement, \( D_{ijkl} \) is an elastic tensor, \( x_{kl}^{ij} \) is the characteristic displacement which is a periodic
function of y. The characteristic displacement has three modes of displacement of the microstructure that reflect the mismatch of the mechanical properties of the constituents and the geometrical configuration of the constituents.

Using the finite element discretization to solve the partial differential equations, the microscopic equation (2) can be written as follows

\[ \left( \int B^T DBdY \right) x = \int B^T D^H dY \]  

where D is the stress strain matrix of the constituents of the composite materials and B is the displacement-strain matrix. \( D^H \) is a vector of column \( ij \) (\( ij = 11,22,12 \)) of the stress-strain matrix \( E \) in 2D. \( \chi \) is the characteristic displacement vector associated with the \( ij \) mode.

The form in equation (3) is very similar to the well known stiffness equation

\[ K \chi = F \]  

the homogenized stiffness matrix can be expressed as follows,

\[ D^H = \frac{1}{|Y|} \int D(1 - B\chi)dY \]  

The force vector used in homogenization has a physical meaning and it is in fact a specific case of initial strain loading. The boundary conditions for the three modes of the characteristic displacement vector \( \chi \) are as follows

For \( \chi^1 \), \( \varepsilon_{11}^0 = 1, \varepsilon_{22}^0 = 0 \) and \( \varepsilon_{12}^0 = 0 \)

For \( \chi^2 \), \( \varepsilon_{11}^0 = 0, \varepsilon_{22}^0 = 1 \) and \( \varepsilon_{12}^0 = 0 \)

For \( \chi^1 \), \( \varepsilon_{11}^0 = 0, \varepsilon_{22}^0 = 0 \) and \( \varepsilon_{12}^0 = 1 \)

Thus it is observed that for 2D problems, by considering three loading cases it is possible to find the homogenized stiffness matrix for the composite material. In practice, after discretizing the domain of the unit cell, it is sufficient to run the finite element program for different unit strain cases.

3. BOUNDARY CONDITIONS OF THE UNIT CELL

To determine the appropriate boundary conditions, we reconsider the main assumption of homogenization theory, i.e. periodicity. The microscopic displacement field \( \chi \) is the Y-periodic solution of equation (4). The unit cell \( Y \) is illustrated in Fig.2 (a,b).

From the definition of periodicity it follows that

\[ x(y_1^0, y_2^0) = x(y_1^0 + Y_1, y_2^0) = x(y_1^0 + Y, y_2^0 + Y_2) \]

\[ = x(y_1^0, y_2^0 + Y_2) \]  

which states that the microscopic displacements of the corner points of the base cell are equal.

Also according to the periodicity the numbers of pairs of nodes located on the opposite edges of the cell can be linked so that opposite edges have identical deformed shapes. For 2D problems the homogenization equation must be solved three times with three types of boundary conditions. The three cases are as follows

Case a: The loading to be imposed in this case is a unit initial strain in \( y_1 \) direction i.e. \( \varepsilon_{11}^0 = 1, \varepsilon_{22}^0 = 0 \) and \( \varepsilon_{12}^0 = 0 \).

To prevent rigid body motion, the horizontal and vertical displacements of the bottom left corner, and the vertical displacement for the bottom right corner are restricted. The loading condition and the symmetry with respect to \( s_2 \) of the base cell lead to the following boundary condition

\[ u(y_1^0, y_2) = -u(y_1 + Y_1, y_2) \]  

\[ v(y_1^0, y_2) = v(y_1 + Y_1, y_2) \]  

The previous two equations mean that the horizontal displacements of the points on the left or right edge have the same displacement of the corresponding point on the other edge but in the opposite direction. And the vertical displacements have the same value and direction.

Similarly from symmetry with respect to \( s_1 \) it follows that

\[ u(y_1, y_2^0) = u(y_1, y_2^0 + Y_2) \]  

\[ v(y_1, y_2^0) = v(y_1, y_2^0 + Y_2) \]  

Case b: The loading condition in this case is \( \varepsilon_{11}^0 = 0, \varepsilon_{22}^0 = 1 \) and \( \varepsilon_{12}^0 = 0 \). As with case (a), because of the symmetry of geometry and loading, we conclude the same boundary conditions.

Case c: The loading condition in this case is a unit initial shear strain \( \varepsilon_{11}^0 = 0, \varepsilon_{22}^0 = 0 \) and \( \varepsilon_{12}^0 = 1 \). In this case the anti-symmetry condition exists. So the displacement of the opposite edges of the base cell may be expressed as:

\[ u(y_1^0, y_2) = u(y_1 + Y_1, y_2) \]  

\[ v(y_1^0, y_2) = -v(y_1 + Y_1, y_2) \]  

\[ u(y_1, y_2^0) = -u(y_1, y_2^0 + Y_2) \]  

\[ v(y_1, y_2^0) = v(y_1, y_2^0 + Y_2) \]  

4. THE HOMOGENIZED STIFFNESS MATRIX

The fiber-reinforced concrete (FRC) can be considered
as a composite made by a matrix of concrete with a brittle behavior and by steel short fibers randomly distributed in the matrix with a volume fraction between 0.5% and 3%. The aim of such reinforcement is to improve the mechanical behavior of the material when tensile forces are acting. In fact, in this way, a limit tensile strength with ductile behavior is ensured to the material.

If the material is loaded by a uniaxial strain, it is possible to observe the following characteristic phases of the behavior of the material:

(I) Initial linear elastic phase. It is characterized by an elastic modulus and the peak strength of the concrete. In this phase, the fibers do not affect the overall material behavior because of their very low volume fraction, and the stress is supported only by the matrix until the tensile strength of the concrete is reached. In this phase the concrete and steel are considered elastic so that the homogeneity can easily established and homogeneous elastic stiffness matrix can be formed.

(II) Crack developing phase. It is characterized by a strain field greater than the limit strain of the matrix; in this case, the composite changes configuration. As a matter of fact, cracks develop in the matrix and the resulting stress is sustained by the composite material made of the cracked concrete and the fibers. In this study, the smeared crack model is considered to model the concrete element after cracking. To establish homogeneity of the composite material after cracking, the unit cell is considered to be subjected to a direct tensile force. The nodes on the side boundaries are subjected to incremental tensile latitude displacements of 0.001 mm.

(III) Failure phase. It is characterized by failure phenomena of the overall composite. Indeed, the value of the stress at damaged conditions, represents the ultimate value of stress for strains greater than ultimate strain: As a matter of fact, by increasing the applied strain, several damage phenomena can occur: slipping between fiber and matrix or plastic compressive strains in the matrix.

To apply the homogenization technique we need to discretize the unit cell to several elements by creating a regular mesh model. The fiber elements are treated as bar elements and the concrete elements are treated as triangle elements. The homogenized stiffness matrix $D^H$ is a symmetric matrix and has the form of equation (15).

$$D^H = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}$$

According to the symmetry of the pattern the values of $D_{11}$ and $D_{22}$ are equals. So it has three parameters which are $D_{11}$, $D_{12}$, and $D_{33}$. For the elastic homogenized matrix, it is easy to calculate according to the loading and boundary conditions discussed in the previous section.

In compression the concrete is assumed to be elastic so the elastic homogenized matrix can be used in all compression elements in the macro-scale domain.

In the elements subjected to tensile stress, the tensile cracks appear after the concrete reaches to the yield tensile stress. After cracking starts the homogenized stiffness matrix changes due to the change of the material properties. The Young's modulus of concrete decreases rapidly when the concrete element starts to crack. In this stage we have to estimate the Young's modulus of each element of concrete after cracking. To do that, the smeared crack model for plain concrete is used. In this approach the cracked concrete is assumed to remain a continuum. Cracks are smeared over the whole area by reducing the material stiffness (Young's and shear modulus). The Young's modulus $E$ in the direction perpendicular to the cracks direction is reduced to the value of $E(1-\omega)$ which represents the damage of the element, where $\omega$ represents the damage factor and $0 \leq \omega \leq 1$. Also, a reduced shear modulus $\alpha G$ is assumed on the cracked plane to account for the aggregate interlocking. The value $\alpha$ is a preselected constant such that $0 \leq \alpha \leq 1$. The stress strain relationship becomes as follows:

$$\sigma_{11} = \begin{bmatrix} 1 & -v & 0 \\ -v & E(1-\omega) & 0 \\ 0 & 0 & 1/\alpha G \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

where, $\omega$ represents the damage of concrete. The poisson's ratio $\nu = 0.3$ and it is assumed to be constant before and after cracking.

The stress strain relation and $\omega$-strain relation used in this study are derived by Bolander [1992] and S.Hiranaka [2002] as shown in Fig.3. According to their work the value of $\omega$ is calculated from equation (17)

$$\omega = 0 \quad \left(\varepsilon_1 \leq \varepsilon_q\right)$$

$$\omega = 1 - \frac{f_t}{E\varepsilon_{11}} \exp\left(\frac{k}{\varepsilon_0} \left(\varepsilon_1 - \varepsilon_q\right)\right) \quad \left(\varepsilon_1 \geq \varepsilon_q\right)$$

where $f_t$ and $\varepsilon_q$ are the maximum elastic tensile stress and strain of concrete respectively. $\varepsilon_0$ is the strain when the element becomes perfectly damaged, and $k$ is an empirical constant to express the strain softening and its
For the accurate calculation of the stiffness matrix, the unit cell should be solved in every integration point of the macro-scale domain in which the neighbor elements started to crack. But this way costs very much of calculation time and large memory size of computers.

To simplify the solution, the unit cell is solved to get the homogenized stiffness matrix after cracking using a uniaxial incremental displacement along $s_1$. The damaged factor $\omega$ and the new Young's modulus $E(1-\omega)$ is calculated. Then the previous homogenization technique is applied to obtain $D^H$. By this way we will have two homogenized matrix, one for the elements before cracking and another one for the elements after cracking.

Using the homogenization method, $D^H$ is estimated for Patterns P1 and P2. Fig.4 shows the values of $D^H$ parameters for pattern P1 with respect to the tensile strain. The width of the cell is 40mm and the length of the fiber is 30mm. the fiber volume content of this example equals 2%. In the elastic range before crack the values of $D_{11}$, $D_{12}$ and $D_{33}$ are constant. These values change rapidly after cracking to another constant value. This means we can define two values for every parameter. From the figure after cracking $D_{12}$ almost vanishes, so the value of $D_{12}$ is set equal to zero. $D_{11}$ and $D_{22}$ are equals before and after cracking. $D^H$ after cracking will be in the form:

$$D^H = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \quad (18)$$

Fig.5 represents a comparison between patterns P1 and P2 for the value of $D_{11}$. The value of $D_{11}$ for pattern P1 is higher than the corresponding value of P2 after cracking. It means that the pattern P1 have more toughness than pattern P2. This may be explained as the existence of two perpendicular fibers in the middle part of pattern P1 gives the fibers more ability to bridge the cracks which occurs in the middle part of the cell. Therefore the number of the damaged elements in the middle part of the unit cell reduced which means more ductility to the material. From the other hand the pattern P2 is weaker in the middle part and the chance for the cracks to grow from the middle is higher than pattern P1 especially in the diagonal direction.

Fig.6 represents the value of $D_{11}$ for pattern P1 using different ratios of fiber content. As expected the higher fiber content the higher stiffness. However the rate of increasing stiffness due to the increase of fiber content after cracking is not constant. The rate decreases by increasing of the fiber content because the area of cracked elements doesn't decrease by the same rate of the increase of the fiber content.
relationship for $\sigma_{11}-\varepsilon_{11}$ can be simplified as shown in Fig.7. After the composite element starts to crack the strength of the element decreases rapidly but it starts to recover again to carry some stress due to the existence of fibers. $\varepsilon_{c}$ represents the tensile strain of concrete, $\varepsilon_{p}$ represents the strain at which the fibers recover the strength of the matrix and $\varepsilon_{u}$ is the max strain after which the element is considered totally damaged. Based on Fig.4, these values are considered as follows; $\varepsilon_{c} = 0.0003$, $\varepsilon_{p} = 0.0005$ and $\varepsilon_{u} = 0.002$.

The values of $D_{11}$, $D_{12}$ and $D_{33}$ before and after cracking should be determined by solving the unit cell using the homogenization method taking into consideration changing the dimensions of the unit cell according to the fiber content, the length, and the diameter of the fibers with keeping the shape of the pattern.

![Fig.8 Dimensions of the test model](image)

The total width of the square unit cell $L$ is decided according to the volume fraction of fibers in the cell and it can be calculated from the following equation

$$L = \sqrt{\frac{A_{f}}{\rho}}$$

(19)

where $A_{f}$ is the total area of fibers in the unit cell and $\rho$ is the volume fraction of fibers.

In this nonlinear analysis for the macroscopic structure, the total load applied is calculated from a series of small displacement increments at the point of loading. At the completion of each incremental solution, the stiffness matrix of each element is adjusted if the element starts to damage, i.e. if the tensile strain in the element exceeds the elastic tensile strain $\varepsilon_{c}$.

The results of the numerical experiment are compared with the experimental results done by J. Yin et al (2003) as shown in Fig.10. In the figure, the experimental results are denoted by the symbol Exp. FE-P1 is the numerical result for pattern P1, FE-P2 is the numerical result for pattern P2 and SF is the volume fraction for 400x100x100 mm with 4-points loading as shown in Fig.8 is analyzed. Short steel fibers 30mm long and 0.6mm diameter are distributed with volume contents 0.0, 0.5 and 1.0% inside the specimen. The results are obtained using the two patterns of the unit cell. Finite element model of the macro-structure has been created as shown in Fig.9 with 1701 nodes and 3200 triangle elements. The microstructure model of pattern PI and P2 are shown in Fig.2(a,b).
steel fibers. From the figure, it is clear that the pattern P1 gives more ductility to the material than the pattern P2. The reason for that is explained in section 4. Fig.10 shows also that the pattern P1 gives a better agreement to the experimental work than pattern P2. However the experimental work has more ductility than numerical model before failure. The reason for that can be the increasing number of long cracks which makes the structure less homogeneous and the error of approximating the real distribution of fibers to a periodic one increases. This may put a limitation to predict the deflection at failure by the homogenization method. However the method can predict most of the tensile behavior with enough accuracy.

Generally from Fig.10, it can be said that the fibers do not considerably influence on the flexural strength of concrete. However, the mean feature of steel fibers is that it can increase the toughness (energy absorption) and enhance the ductility of concrete as shown from the energy absorption-strain curve in Fig.11 based on the results of pattern P1. The enhanced behavior of steel fiber reinforced concrete over its unreinforced counter-parts comes from its improved capacity to absorb energy during fracture. While a plain unreinforced matrix fails in a brittle manner at the occurrence of cracking stresses, the fibers in fiber reinforced concrete continue to carry stress beyond matrix cracking, which helps maintain structural integrity and cohesiveness in the material.

As an application for the introduced modeling, consider the structure shown in Fig.12. For the macro-scale structure 2000 points are used to create the mesh model using the 2D Delaunay triangulation system with 3759 triangle elements. The pattern P1 is used for the unit cell. The load P (KN) is applied at the free end. An incremental vertical displacement of 0.001 mm is used as a boundary condition at the point of loading. Two fiber volume contents 1% and 2% are used. The results in Fig.13 show the load-deflection relationship for the numerical example. As expected the short steel fibers inclusions remarkably increase the concrete toughness. The distribution of principal tensile stress is shown in Fig.14 at peak and at failure. The failure occurs when the cracks go through the critical section and separate the structure into two pieces.

Fig.10 The load–deflection relation for SFRC with different fiber contents

Fig.11 The energy absorption for the computed SFRC models

Fig.12 Dimensions of the numerical Model

Fig.13 The load–deflection relation for the numerical model
REFERENCES


6. CONCLUSIONS

Homogenization method is used to model SFRC as a composite material. A periodic distribution of steel fibers has been considered. Two patterns P1 and P2 for the unit cell are examined. The homogenized stiffness matrix for the composite is determined before and after cracking. A FORTRAN computer program is developed to solve to homogeneous equations. The following results have been obtained;

- The homogenized parameters $D_{11}$, $D_{22}$, $D_{33}$ have constant values before element cracking and it reduces rapidly after cracking to another constant value, the parameter $D_{12}$ tends to zero after cracking.
- The Pattern P1 gives higher strength than pattern P2 because its higher ability to strengthen the middle part of the unit cell.
- The results show that the method gives high agreement with the experimental work in the elastic range. This agreement decreases by the increasing of cracks because the exceeding of cracks makes the material less homogenized. However the introduced modeling can predict the tensile behavior with enough accuracy
- The pattern P1 gives better results than the pattern P2 compared to the experimental results.
- The homogenization method facilitates modeling the SFRC structures regardless the locations of the fibers which are distributed randomly.

For future work, extension the method to 3D with a 3D unit cell is necessary. Also the study will be extended to study the effect of the shape, length and diameter of the fibers on the behavior of SFRC.