An Extended Goodwin Model with International Trade

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1 Introduction

Chaos is one of the most interesting topics in recent economic dynamics. Pohjola (1981) constructed a simple discrete-time model that is capable of generating chaotic growth paths by modifying Goodwin's (1967) growth cycle model. His modification is the introduction of a bargaining equation. Takamasu (1997) indicated by computer simulations that chaotic movements are observed if Skott's output expansion function (See Skott (1989)) replaced the assumption of fixed capital-output ratio[1].

In this paper we demonstrate the emergence of chaos by allowing for international trade to the Goodwin model. Although it involves some 'bold' assumptions, it is meaningful to construct a framework of Goodwin model with international trade, because it seems no work has discussed this problem.

The plan of this paper is as follows. We set up our model in section 2, then long-run equilibria of the model are considered in section 3. Dynamic properties of the model are analyzed in sections 4 and 5. The final section summarizes the results of the extension.

2 The Model

We consider a two-nation two-good model, in which we assume a simple horizontal trade. We make the following assumptions.

(A1) There are two countries that produce the different commodities.

(A2) Each commodity requires for its production capital and labour as input.

(A3) There is no mobility of both productive factors between two countries.

(A4) Capitalists save all profits, and automatically invest them. On the other hand, workers consume the whole wages.

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1) Although the same assumption and result have already been shown in Goodwin (1990), it has not referred to Skott's (1989) equation.

(A5) The utility of a consumer is maximized at a positive combination of two goods. Thus a horizontal trade continues permanently.

(A6) The change in the money wage rate is determined by a Phillips curve, which is common between two countries.

(A7) The capital-output ratio is constant.

(A8) We rule out the trend. The rate of population growth and technical progress are both equal to zero.

(A9) The rate of capital depreciation is a positive constant and common between countries.

(A10) Two countries use the same currency.

Although assumption (A9) simplifies the argument, it would not change the results much.

From these assumptions, we can represent our model as six algebraic equations.

$$K_i(t+1) = (1-\delta)K_i(t) + (1-u_i(t))Y_i, \tag{1}$$

$$w_i(t+1) = \{1 + f(v_i(t))\}w_i(t), \tag{2}$$

$$Y_i(t+1) = \min\left(a_i N_i, \frac{K_i(t+1)}{\sigma_i}\right),^{2)} \tag{3}$$

$$L_i(t+1) = \frac{Y_i(t+1)}{a_i}, \tag{4}$$

$$v_i(t+1) = \frac{L_i(t+1)}{N_i}, \tag{5}$$

$$u_i(t+1) = \frac{w_i(t+1)L_i(t+1)}{p_i Y_i(t+1)}, \tag{6}$$

where the symbol K_i means the capital stock in country i, Y_i is the output level, w_i the money wage rate, v_i the employment ratio, N_i the labour available, L_i the employed labour, and u_i the labour share. On the other hand, the parameter δ means the depreciation rate, a the labour productivity, and σ the capital-output ratio.

Concerning assumption (A5), we define a utility function, which shows the preference of the representative workers in country 1 as indifference curves in Fig. 1 and formally as follows.

2) Here $\min(x, y)$ is a function which assigns the lower value between x and y.

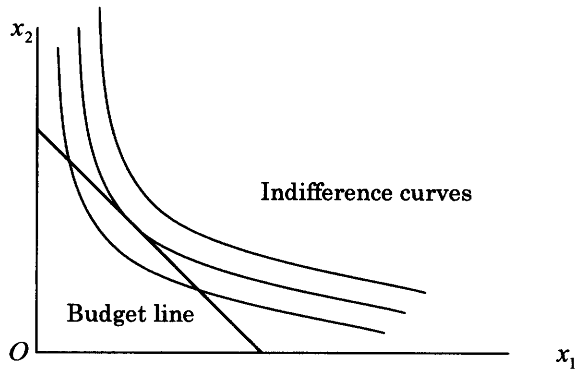

Fig. 1

$$U_1 = x_1^{\theta_1+\rho}\, x_2^{(1-\theta_1)\rho^{-2}}, \tag{7}$$

where θ_1 is a parameter, greater than zero and smaller than unity; the symbol ρ means the price proportion p_2/p_1, and the variable x_1 means the amount of consumption good i which country 1 consumes. This function corresponds to a special Cobb-Douglas function. It reflects changes in the price proportion. That is, the utility function contains a *bias* with respect to goods produced in different countries.

From this function, we can derive an optimal combination for country 1 on a budget line for given price vector.

$$(x_1, x_2) = \left(\frac{(\theta_1 p_1 + p_2)w_1 L_1}{p_1(p_1 + p_2)}, \frac{(1 - \theta_1)p_1 w_1 L_1}{p_2(p_1 + p_2)}\right). \tag{8}$$

This is the demand function for given a price vector. It should be noted that country 1 consumes the domestic goods by at least θ_1 of his income.

Similarly in country 2, the utility function and the choice vector are as follows:

$$U_2 = x_1^{(1-\theta_2)\rho^2}\, x_2^{\theta_2+\rho^{-1}}, \tag{9}$$

$$(x_1, x_2) = \left(\frac{(1 - \theta_2)p_2 w_2 L_2)}{p_1(p_1 + p_2)}, \frac{(p_1 + \theta_2 p_2)w_2 L_2)}{p_2(p_1 + p_2)}\right). \tag{10}$$

These show the preference and the consumption behaviour of country 2. Thus, we have the following price adjustment equations with the excess demand functions on the right-hand-side.

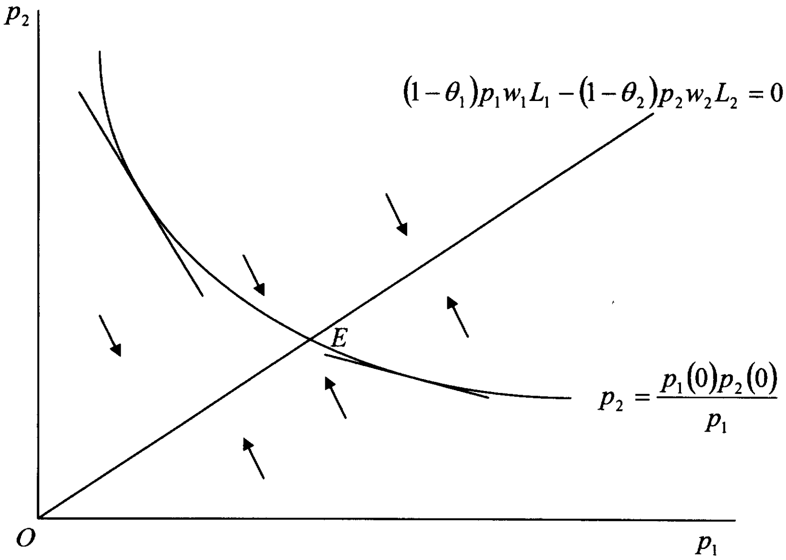

Fig. 2

$$\dot{p}_1 = -\{(1-\theta_1)p_1w_1L_1-(1-\theta_2)p_2w_2L_2\}, \tag{11}$$

$$\dot{p}_2 = (1-\theta_1)p_1w_1L_1-(1-\theta_2)p_2w_2L_2. \tag{12}$$

As shown in Fig. 2 the short-run equilibrium (i.e., balance of trade) price proportion $\rho_s^* = (1-\theta_1)w_1L_1/(1-\theta_2)w_2L_2$ is stable. When the parameter θ_1 is equal to θ_2[3] the slope of a local change in price vector $\boldsymbol{p}$ is expressed as a tangent of a hyperbolic curve determined by an initial vector $\boldsymbol{p}(0)=(p_1(0), p_2(0))$, while the length of the vector $\dot{\boldsymbol{p}}$ becomes smaller when the point comes near a line $(1-\theta_1)p_1w_1L_1-(1-\theta_2)p_2w_2L_2=0$, which means the *equilibrium ray*. If the adjustment is sufficiently flexible, the trade between two countries at period t is conducted at an equilibrium $E=\left(\sqrt{\frac{p_1(0)p_2(0)w_2(t)v_2(t)N_2}{w_1(t)v_1(t)N_1}}, \sqrt{\frac{p_1(0)p_2(0)w_1(t)v_1(t)N_1}{w_2(t)v_2(t)N_2}}\right)$, which is a positive intersection point between a hyperbolic curve and the equilibrium ray. Besides, in the discretized model we consider two more cases.

One case is when the tatonnement process is closed before a price vector reaches a equilibrium ray. Then the commodity corresponding to an excess supply is left and freely disposed. We allow for such a disequilibrium in the short term. The other is the case where

3) For a while we continue our argument holding this condition for simplicity.

a trading price vector is determined using the current price ratio $(p_1(t)/p_2(t))$. That is an equilibrium point is calculated as the intersection point between the hyperbolic curve through the current price vector and the equilibrium ray. These problems are discussed in the rest of this section.

For the case of sufficient flexibility, we summarize our model as the following six equations. We call it System 1.

System 1

$$K_i(t+1) = \left\{1 - \delta + \frac{1-u_i(t)}{\sigma_i}\right\} K_i(t), \tag{1'}$$

$$w_i(t+1) = \{1 + f(v_i(t))\} w_i(t), \tag{2}$$

$$v_1(t+1) = \min\left(\frac{K_i(t+1)}{a_i \sigma_i N_i}, 1\right), \tag{5'}$$

$$p_1(t+1) = \sqrt{\frac{p_1(0)p_2(0)w_2(t+1)v_2(t+1)N_2}{w_1(t+1)v_1(t+1)N_1}}, \tag{13}$$

$$p_2(t+1) = \sqrt{\frac{p_1(0)p_2(0)w_1(t+1)v_1(t+1)N_1}{w_2(t+1)v_2(t+1)N_2}}, \tag{14}$$

$$u_i(t+1) = \frac{w_i(t+1)}{a_i p_i(t+1)}. \tag{6'}$$

This system includes five sets of variables with $i = 1, 2$: K_i, w_i, v_i, u_i, and p_i.

On the other hand, when the adjustment is not flexible in the discretized version the dynamic system becomes more complex.

We rewrite the tatonnement process expressed eqs. (11) and (12) as simultaneous difference equations.

$$p_1(t+1) = [1 - \alpha\{(1-\theta_1)p_1(t)w_1L_1 - (1-\theta_2)p_2(t)w_2L_2\}]p_1(t), \tag{15}$$

$$p_2(t+1) = [1 + \alpha\{(1-\theta_1)p_1(t)w_1L_1 - (1-\theta_2)p_2(t)w_2L_2\}]p_2(t), \tag{16}$$

where the parameter α means an adjustment speed. Hence eqs. (13) and (14) in the flexible dynamic system are rewritten using somewhat more general functional forms as follows:

$$p_i(t+1) = h_i(\alpha, \tau, p_1(t), p_2(t), w_1(t+1), w_2(t+1), v_1(t+1), v_2(t+1)) \tag{17}$$

where parameter τ means the number of iterations, and h_i is a function derived from the tatonnement process expressed by eqs. (15) and (16). Note that when $\theta_1 = \theta_2$ the following equations hold.

$$\lim_{\tau\to\infty,\alpha\to 0} h_1 = \sqrt{\frac{p_1(0)p_2(0)w_2(t+1)v_2(t+1)N_2}{w_1(t+1)v_1(t+1)N_1}},$$

$$\lim_{\tau\to\infty,\alpha\to 0} h_2 = \sqrt{\frac{p_1(0)p_2(0)w_1(t+1)v_1(t+1)N_1}{w_2(t+1)v_2(t+1)N_2}}.$$

We describe System 2 by replacing eqs. (13) and (14) with eq. (17). The system may involve disequilibrium states.

System 2

$$K_i(t+1) = \left\{1 - \delta + \frac{1-u_i(t)}{\sigma_i}\right\} K_i(t), \tag{1)'$$

$$w_i(t+1) = \{1 + f(v_i(t))\} w_i(t), \tag{2}$$

$$v_i(t+1) = \min\left(\frac{K_i(t+1)}{a_i\sigma_i N_i}, 1\right), \tag{5)'$$

$$p_i(t+1) = h_i(\alpha, \tau, p_1(t), p_2(t), w_1(t+1), w_2(t+1), v_1(t+1), v_2(t+1)), \tag{17}$$

$$u_i(t+1) = \frac{w_i(t+1)}{a_i p_i(t+1)}. \tag{6)'$$

Note that both Systems 1 and 2 contain the Goodwin model with no trend as a special case in which all parameters and initial values are same between two countries[4]. In the next section, we consider the existence of long-run equilibria of those two systems.

3 Equilibria of Systems

Now we show the existence of long-run equilibria of Systems 1 and 2. From eq. (1)′,

$$-\delta + \frac{1-u_i^*}{\sigma_i} = 0, \text{ thus}$$

$$u_i^* = 1 - \delta\sigma_i. \tag{18}$$

4) In this case the price vector is fixed at an initial position.

When the labour share is equal to u_i^*, the net investment of capital in country i is zero. Eq. (2) determines the equilibrium employment ratio v_i^* so that the Phillips function $f(v)$ equals zero[5]. From the property of the function $f(v)$, v_i^* should be smaller than unity. From eq. (5)′ we obtain

$$K_i^* = a_i\sigma_i v_i^* N_i.$$

The equilibrium values of w_i^* and p_i^* are different between Systems 1 and 2. In the case of System 1, solving the following simultaneous equations gives these values.

$$(p_1^*)^2 = \frac{p_1(0)p_2(0)w_2^*N_2}{w_1^*N_1},$$

$$(p_2^*)^2 = \frac{p_1(0)p_2(0)w_1^*N_1}{w_2^*N_2},$$

$$w_i^* = a_i p_i^* u_i^*.$$

On the other hand, concerning the case of System 4.2 we can obtain an equilibrium price proportion. In this case the equilibrium condition of the international trade is described as

$$(1-\theta_1)p_1w_1L_1 - (1-\theta_2)p_2w_2L_2 = 0. \tag{19}$$

Then the long-run equilibrium price proportion ρ^* is derived as eq. (20) below by substituting the following equation to eq. (19)

$$w_i = a_i u_i^* p_i.$$

$$\rho^* = \sqrt{\frac{(1-\theta_1)a_1u_1^*N_1}{(1-\theta_2)a_2u_2^*N_2}}. \tag{20}$$

When parameter θ_1 equals θ_2 the long-run equilibrium price proportion corresponds to that of System 1. However the equilibrium values w_i^* and p_i^* are different from those in System 1 because they depend on the initial condition $\boldsymbol{p}(0)$ and the parameters α and τ.

4 The Mechanism of Dynamics

From the above arguments it is shown that Systems 1 and 2 have a meaningful equilibrium point for real variables. Next, we consider the common mechanism of dynamic process

5) From assumption (A6) equilibrium employment rates v_i^*, $i = 1, 2$, are equal. It can simplify the equations ensuing.

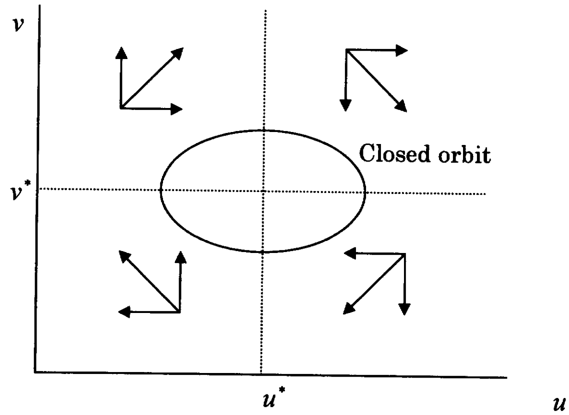

Fig. 3

between two systems.

When the labour share of the first country is greater than u_1^* and the employment ratio is also greater than v_1^*, it is clear that the main economic variables K_1, Y_1, L_1, and v_1, except the money wage rate, all fall down because the pressure on the entrepreneurs' profits weakens the investment activities. The wage rate continues to rise until v_1 gets smaller than or equal to v_1^*. This property is the same as Goodwin's as shown in Fig. 3.

However, the assumption concerning international trade makes the behaviour of the labour share u_1 unclear. In fact in the case of System 1 it depends on many variables: K_i, w_i, v_i, and u_i with $i = 1, 2$, while in the case of Goodwin model it depends on $v_1(t)$[6]. Similarly the nominal capital stock or the nominal domestic product also changes by the influence of the other country's behaviour. For example, if the foreign workers demand the domestic products so that the rise of its price can offset the growth of money wage, the domestic labour share decreases. Thus we expect that trajectories generated by our dynamic equations are more complex than that of Goodwin model without trade represented in Fig. 3.

In the next section, we examine the characteristics and properties of Systems 1 and 2 with some numerical examples.

5 Computer Simulations

We confirm by computer simulations whether our conjectures are realized. Moreover we

6) See Goodwin (1967).

examine the stability of the long-run equilibria. The analysis consists of two parts. That is, first we examine the simple case such as the same capital-output ratio $\sigma = \sigma_1 = \sigma_2$ in System 1 and System 2. Next we simulate those systems in more general case with the different capital-output rates.

Before proceeding to the analysis, here we add a constraint and rewrite two functions, that is, eqs. (1)′ and (2) in our models in order to enhance their reality. These would not change much our results, and they make the ranges of variables more realistic. First, the wage rates in both countries are constrained so that the expected entrepreneurs' real profits rates is greater than or equal to a constant rate, that is, the minimum rate of real profits. (The last price vector determines the expected rates of profit.) We fix the rate as $R_{\min} = 0.03$. For simplicity, the rate is common between countries. On the other hand, we allow the capitalists' consumption. Assumption (A4) is modified as follows: capitalists save s of their profits, while workers consume all wages. The parameter s means a constant saving ratio. It is assumed $s = 0.6$.

Hence rewritten equations are as follows:

$$w_i(t+1) = \min\left[\{1 + f(v_i(t))\}w_i(t),\ a_i(1 - \sigma_i R_{\min})p_i(t)\right], \tag{21}$$

$$K_i(t+1) = \left\{1 - \delta + \frac{s(1 - u_i(t))}{\sigma_i}\right\} K_i(t). \tag{22}$$

Next we rewrite the Phillips function so that a computer can calculate it.

$$f(v) = \frac{9.638 \times \exp\left[\log\{100(1.001 - v) \times (-1.394)\}\right] - 9.0}{100} \tag{23}$$

From this equation, the equilibrium employment ratio is calculated by

$$v^* = 1.001 - \exp\left(\frac{\log 9.638 - \log 9}{1.394} - \log 100\right).$$

In this case the equilibrium employment ratio is $v^*(= v_1^* = v_2^*) = 0.94621047$, and it is smaller than unity.

In our first analysis parameters are fixed as follows: $\delta = 0.04$, $\sigma_1 = 3.5$, and $\sigma_2 = 3.5$, while both parameters a_1 and a_2 are determined by initial conditions as

$$a_i = K_i(0)/\sigma_i v_i(0) N_i. \tag{21}$$

On the other hand, the initial conditions are as follows: $K_1(0) = K_2(0) = 1$; $N_1 = N_2 = 1$; $u_1(0) = 0.6$, $u_2(0) = 0.65$; $v_1(0) = 0.95$, $v_2(0) = 0.97$; $p_1(0) = 1$. Other variables can be derived from them as follows:

$$w_1(0) = a_1 p_1(0) u_1(0),$$

$$p_2(0) = \sqrt{\frac{p_1(0) w_1(0) v_1(0) N_1}{a_2 u_2(0) v_2(0) N_2}},$$

$$w_2(0) = a_2 p_2(0) u_2(0).$$

Now we examine the relationship between the existence of international trade and stability of dynamics in case of $\sigma_1 = \sigma_2$ by using figures below. First we demonstrate that each original Goodwin model, when separated, generates a periodic solution if both the rate of population growth and the technical progress are equal to zero. (When there is no trade between countries, we fix the price vector at the initial position shown above.) Fig. 4 shows the dynamics of country 1 without trade, while the behaviour of country 2 is shown in Fig. 5. The horizontal axis means the labour share, and the vertical axis is the employment ratio in these figures. Both trajectories are cyclical and periodic.

On the other hand, Fig. 6 and Fig. 7 represent respectively the dynamics with trade in the case of System 1 in both countries. The trade is under the perfect price flexibility, i.e., the trade is balanced. System 2 also generates the same results as those of System 1 as shown Figs. 8 and 9. In this system, we assign parameters as , $\theta = \theta_1 = \theta_2$, $\alpha' = \alpha(1 - \theta) = 0.1$, and $\tau = 300$. If the number of periods for adjustment is less than 300, there is a greater possibility that trade between two countries remain unbalanced. The dynamic paths of country 1 in System 2 in the case of $\tau = 12$ is represented in Fig. 10. This figure seems to draw the more stable trajectory than behaviour in Figs. 6-9, though all the figures 6-10 show more complicated behaviour than in Figs. 4 and 5. The reason may be that the shortage in the number for price adjustment weakens the interaction between countries.

Next let us consider the more general case. We assign the different values to the capital-output ratios between two countries. Then with regard to the fluctuations in country 1, three cases are observed in six figures 11-16. In former three figures, we assign $\sigma_2 = 2.6$, while in latter $\sigma_2 = 4.7$. The former and the latter three figures are respectively corresponding to System 1, System 2 with $\tau = 300$, and System 2 with $\tau = 12$. The parameter α' is fixed as $\alpha' = 0.1$.

Figs. 11 and 12 show that the economy reaches a limit cycle in spite of initial chaotic

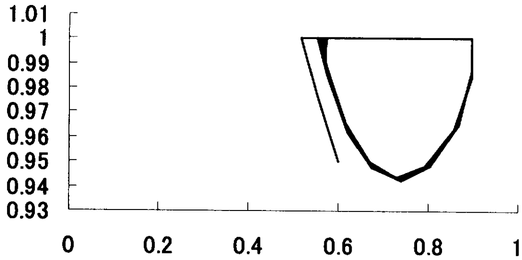

Fig. 4　Country 1 without trade

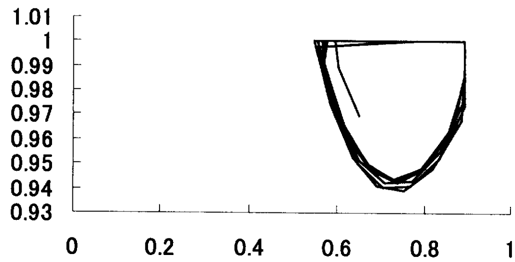

Fig. 5　Country 2 without trade

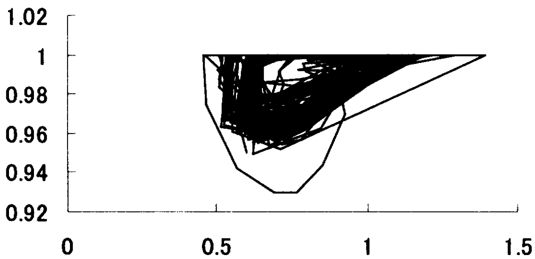

Fig. 6　Country 1 with trade

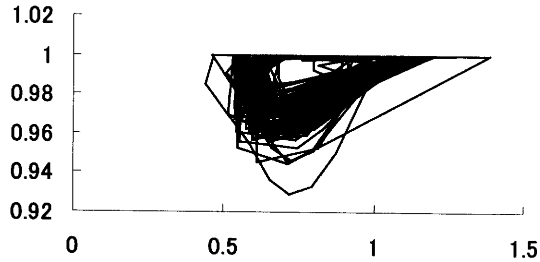

Fig. 7　Country 2 with trade

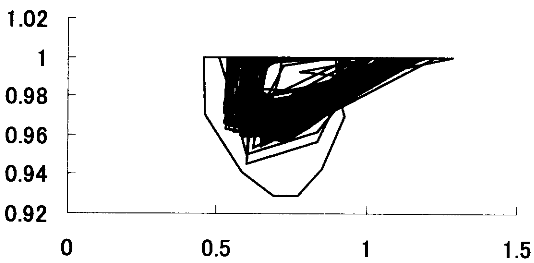

Fig. 8　Country 1 with unbalanced trade

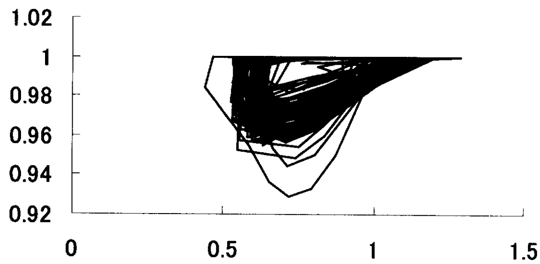

Fig. 9　Country 2 with unbalanced trade

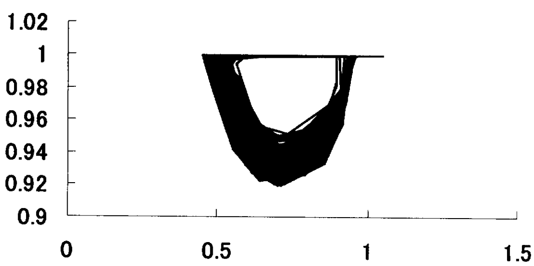

Fig. 10　System 2 with $\tau = 12$

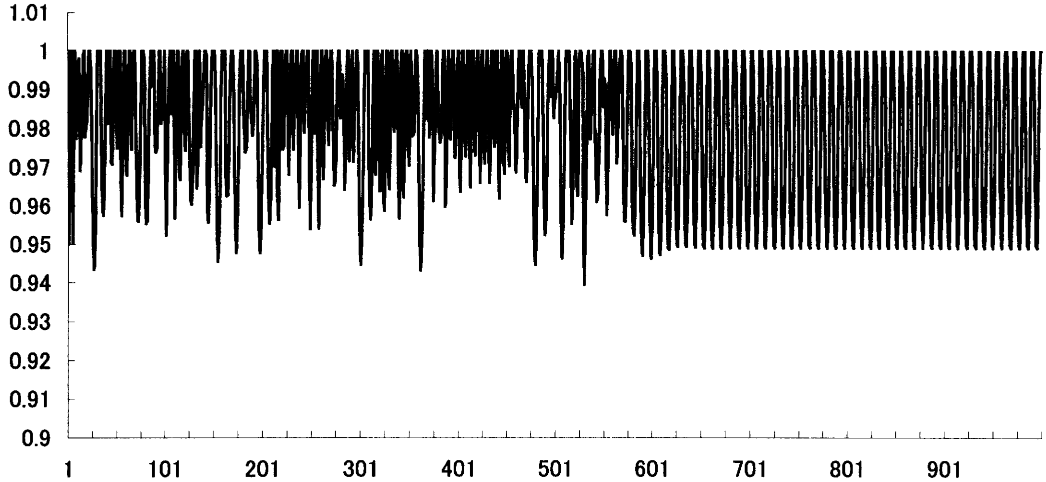

Fig. 11 System 1 ($\sigma_2 = 4.7$)

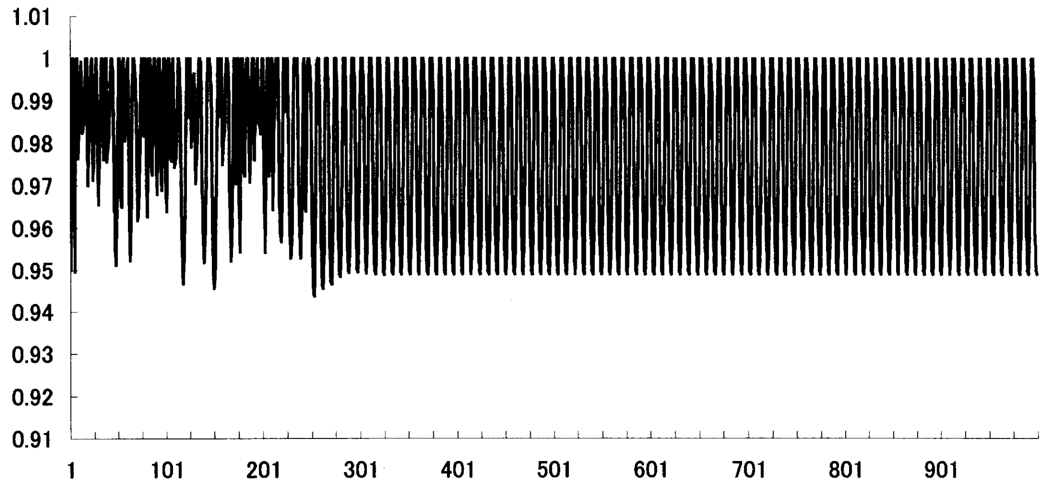

Fig. 12 System 2 ($\sigma_2 = 2.6$, $\tau = 300$)

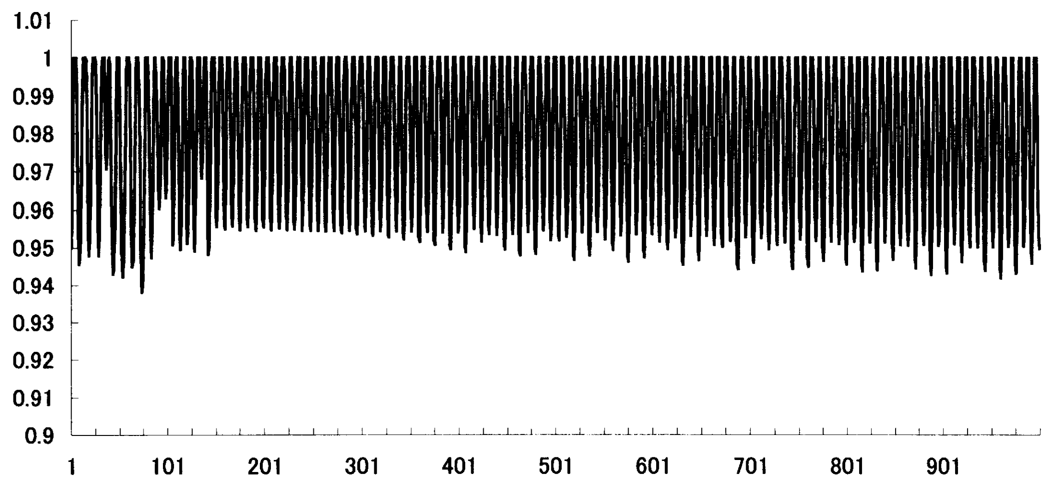

Fig. 13 System 2 ($\sigma_2 = 2.6$, $\tau = 12$)

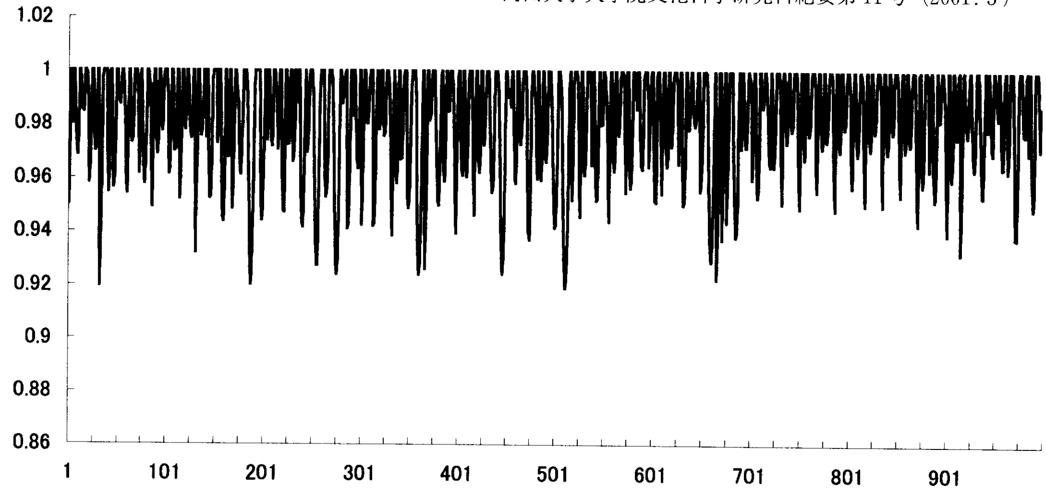

Fig. 14　System 1 ($\sigma_2 = 4.7$)

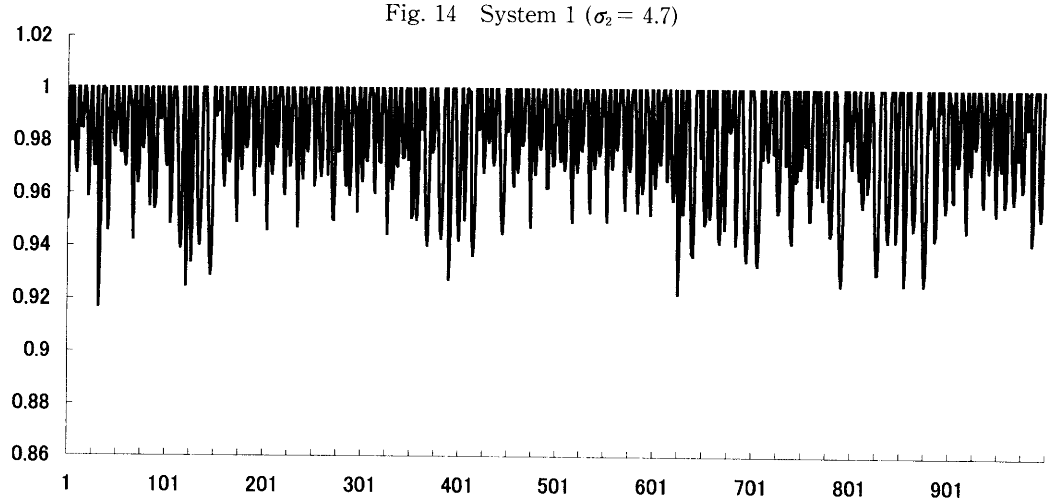

Fig. 15　System 2 ($\sigma_2 = 4.7$, $\tau = 300$)

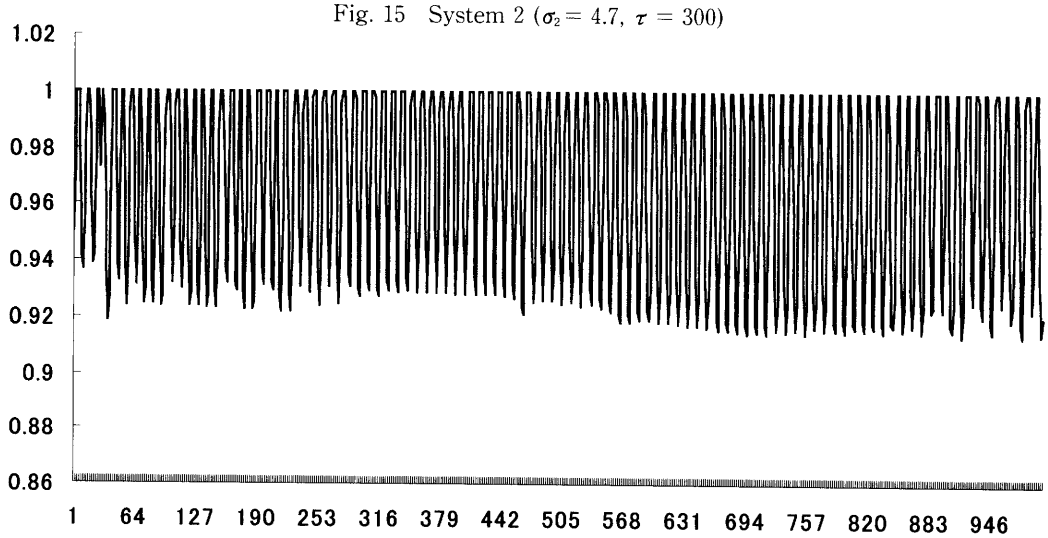

Fig. 16　System 2 ($\sigma_2 = 4.7$, $\tau = 12$)

behaviour. In Figs. 13 and 16, almost periodic orbits are depicted. However these are slightly divergent. On the other hand, dynamic behaviour in Figs. 14 and 15 seems to be chaotic. Moreover we have conducted many simulation analyses using various σ_2. As a result we classify the properties of fluctuation as in Table 1. Except the case in which the effects of the price vector are weaker than others, we can observe chaos. Therefore we conclude that the economy is more chaotic if there is an international trade in spite of the two boundaries by the full employment and the minimum profit rate.

6 Concluding Remarks

In this paper we have extended a Goodwin model by introducing into the model international trade between two countries. First, we confirmed that each original Goodwin model, when separated, generates a periodic solution even if both the rate of population growth and the technical progress are equal to zero. Next, from computer simulations concerning the extended model, we observed that the dynamic behaviour becomes more complicated if trade is allowed. The economy starting from a position near a positive equilibrium continues to fluctuate permanently without converging to an equilibrium point nor to a limit cycle. That is, the behaviour seems to be chaotic.

With respect to this point, we indicated that the effect by the price vector, which is determined in the short-run, makes the behaviour of the labour share unclear in each country. In particular, in the case of the shortage in price adjustment we can not demonstrate any

System 1	System 2 with $\tau = 300$	System 2 with $\tau = 12$	σ_2
Chaotic	Chaotic	Chaotic	2.0
Limit Cycle	Limit Cycle	Slightly Divergent	2.2
Chaotic	Chaotic	Slightly Divergent	2.3-2.5
Limit Cycle	Limit Cycle	Slightly Divergent	2.6
Chaotic	Chaotic	Slightly Divergent	2.7-3.1
Chaotic	Chaotic	Limit Cycle	3.2
Limit Cycle	Limit Cycle	Slightly Divergent	3.3-3.6
Chaotic	Chaotic	Slightly Divergent	3.7, 3.8
Limit Cycle	Limit Cycle	Slightly Divergent	3.9-4.1
Chaotic	Chaotic	Slightly Divergent	4.2-4.7
Limit Cycle	Limit Cycle	Little Divergent	4.8

Table 1

chaotic behaviour. In addition, it is shown by way of illustrations that chaotic variations occur in a subspace where the economy can survive.

When we generalized a Goodwin model, an assumption is altered. That is, we assumed the absence of trend for simplicity of arguments because this would not change much our results. Furthermore in our model there are only two countries. If the trade is more diversified among many countries, the stability/instability property may be affected. These problems are remained to be attacked in the future.

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